A NEW TREATISE

ON

ELEMENTS OF MECHANICS

ESTABLISHING STRICT PRECISION

IN THE MEANING OF

DYNAMICAL TERMS

ACCOMPANIED WITH AN

APPENDIX

ON

DUODENAL ARITHMETIC AND METROLOGY.

BY

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PHILADELPHIA:
PORTER & COATES,
822 CHESTNUT STREET.
PUBLISHED FOR THE AUTHOR.
1875.

WESTCOTT & THOMSON,
Stereotypers and Electrotypers, Phila.

SHERMAN & CO.,
Printers, Phila.

TO

WILLIAM SELLERS, ESQ.,

THIS WORK IS RESPECTFULLY DEDICATED,

AS A MARK OF APPRECIATION OF HIS

DISTINGUISHED ACCOMPLISHMENTS

IN THE MECHANIC ARTS,

BY THE AUTHOR.

PREFACE.

THE principal objects in introducing this new treatise on mechanics are the establishment of strict precision in the meaning of dynamical terms, and the classification of physical quantities into elements and functions.

A revision of the principles of mechanics is a necessity long felt, and frequently acknowledged in the discussions of learned men, who have heretofore disagreed as to the true meaning of technical terms and the constitution of dynamical quantities.

The prevalent discordance on these topics has caused the delay of this publication for over ten years, in which interim various discussions thereupon have been published, both in Europe and in America, indicating the confused condition of the subject.

In scientific periodicals we rarely find a sound article on dynamics, but the action and combination of physical elements are treated as if governed by individual judgment, instead of the fixed and immutable ordinances of nature.

A pamphlet entitled *Principles of Dynamics*, exposing the confusion in the science of dynamics, has been published and presented to the principal libraries and institutions of learning.

In addition to the objects above mentioned, this treatise contains sufficient new and original matter to warrant its publication.

The Appendix is added for the purpose of supplying the student with materials for the coming revision and final establishment of an international system of metrology, which must be attended to sooner or later.

ALPHABETICAL INDEX.

INTRODUCTION.

THIS treatise is written for students of Mechanics, and the technical terms herein adopted are those used in the machine-shop, rejecting the ideal vocabulary heretofore used in text-books and colleges.

In order to establish a standard language in Mechanics, it is hoped that institutions of learning will approve and confirm the terms and distinctions of dynamical quantities as adopted and defined in these pages, 57 to 69, inclusive.

The distinction between the terms *force*, *power* and *work* has heretofore not been clearly defined, but either of these terms has been promiscuously applied to either or all those quantities, according to individual caprices.

Work has thus been distinguished by a variety of terms indicating different characters of that function. It has generally been maintained that work is independent of time, and that power is dependent on time, both of which propositions are incorrect.

REJECTED TERMS.

Energy. This term is used to denote work, but the sense of it conveys an idea of a different virtue—namely, that of activity or vigor, which is power. We say that a man has a great deal of energy when he can accomplish much work in a short time, which is a virtue of power; but if he accomplishes the same quantity of work in a much longer time, we do not give him credit for much energy. The term energy, if employed at all, ought to be applied to power alone; but as we have the expressive term *power* for that function, it is better to dispense with the term energy in dynamics.

The term *work* is the proper name for the function which has been called energy.

Quantity of Motion is a term also used to denote work, which latter is a different function from that of motion. The sense of this term is inseparably associated with an idea of more or less space, which is a function of velocity and time without regard to force; and as force is an element of work, the term *quantity of motion* should be rejected as improper to denote that function.

The words **Actual, Total, Quantity, Mode, Potential, Intrinsic, Kinetic Effort,** etc. are often appended to terms without affecting the *nature* of the quantity so denoted; the objection to which is that one and the same quantity is differently defined according to the combination of these appended words.

As an illustration of the effect of these appendages to terms in dynamics, we may apply them to geometrical quantities; for instance, *volume* in geometry corresponds to *work* in dynamics, and may be expressed thus:

Volume of a cube.
Cubical volume of a sphere.
Total intrinsic volume of a cone.
Actual potential volume of a cylinder.
Total intrinsic quantity of volume of a pyramid.
The actual total quantity of voluminous cubic inches in an intrinsic cubic foot is 1728. (A cubic foot is 1728 cubic inches.)

All these expressions mean simply *volume;* as the different combinations of terms denoting work mean simply work, or the product of the three simple elements **force, velocity** and **time.**

The rejection of the superfluous terms will render the subject of dynamics much easier to teach, learn and remember.

There is also an expression generally used in the English language —namely, "*Consumption of coal per horse-power per hour,*" which is not correct, or rather it is nonsense. The intended idea should be expressed, "*Consumption of coal per hour per horse-power.*" It is the fuel which is divided by time, and not the power.

The consumption of fuel is work, which divided by time is power.

There exists no such quantity in dynamics as power per time, but power multiplied by time is work, and work per time is power.

The following list shows which terms are herein rejected and adopted:

DYNAMICAL TERMS.

Rejected Terms.	Reason for Rejection.
Effort of force.	Means simply force.
Efficiency of force.	" " "
Acting force.	All forces act.
Force of motion.	Means motive force.
Working force.	" " "
Quantity of moving force.	" " "
Quantity of motion.	Has no definite meaning.
Mode of motion.	" " "
Mode of force.	" " "
Moment of activity.	Means simply power.
Mechanical power.	" " "
Mechanical effect.	" " "
Quantity of action.	" " "
Efficiency.	" " "
Rate of work.	" " "
Dynamic éffect.	Used for power or work.
Quantity of work.	Means simply work.
Actual total quantity of work.	" " "
Total amount of work.	" " "
Actuated work.	" " "
Vis-viva.	" " "
Living force.	" " "
Energy.	" " "
Actual energy.	" " "
Potential energy.	" " "
Kinetic energy.	" " "
Energy of motion.	" " "
Energy of force.	" " "
Heat a form of energy.	" " "
Heat a mode of motion.	" " "
Mechanical potential energy.	" " "
Quantity of energy.	" " "
Stored energy.	" " "
Intrinsic energy.	" " "
Total actual energy.	" " "
Work of energy.	" " "
Equation of energy.	Formula for work.
Equality of energy.	Primitive and realized work.

ELEMENTS OF MECHANICS.

MECHANICS is that branch of natural philosophy which treats of the three simple physical elements, **force, motion** and **time**; with their combinations constituting **power, space** and **work.**

Mechanics is divided into two distinct parts—namely,

STATICS AND DYNAMICS.

STATICS is the science of forces in equilibrium or at rest: it is subdivided into three branches, treating respectively of solids, liquids and gases.

Statics, strictly speaking, refers to forces in regard to solids.

Hydrostatics treats of the pressure and equilibrium of liquids.

Aerostatics treats of the pressure and equilibrium of air or gases.

DYNAMICS is the science of force in motion, producing power and work; and is also subdivided into three branches, embracing respectively, solids, liquids and gases.

Dynamics, strictly speaking, refers to power and work of solids.

Hydrodynamics treats of the power and work of liquids.

Aerodynamics treats of the power and work of air or gases.

FORCE.

The term **Force** means any action which can be expressed simply by weight, and which can be realized only by an equal amount of reaction. Force is derived from a great variety of sources, but whenever it is simply force it can invariably be expressed by weight, without regard to motion, time, power or work.

A detailed explanation of force is given in Dynamics.

STATICS.

Statics is the science of forces in equilibrium; it embraces the strength of materials, of bridges and of girders; the stability of walls, steeples and towers; the static momentum of levers, with their combination into weighing scales, windlasses, pulleys, funicular machines, inclined planes, screws, catenaria, and all kinds of gearing.

The magnitude and direction of a force can be represented by a straight line; but no force can be realized without an equal amount of resistance in the opposite direction, which likewise can be represented by a straight line.

PROBLEM 1.

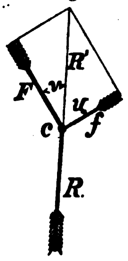
Fig. 1.

Let the line F represent the magnitude and direction of a force acting on a point c at rest or in motion; and let R represent an equal amount of resistance in the opposite direction; then the force F and resistance R are said to be in equilibrium.

When two or more forces act in one or the same direction, as in the case of several men pulling on one rope, the several forces are added together, and the sum is considered as only one force.

PROBLEM 2.

Fig. 2.



Let the lines F and f represent magnitudes and directions of two forces acting at a point c. Complete the parallelogram, and the diagonal R' will represent the magnitude and direction of the consolidated action of the two forces F and f.

The diagonal R' is called the *resultant* of F and f. Make R equal and opposite to R'; then R represents the magnitude and direction of the resistance at the point c. The lengths of the lines F, f and R in any unit of measure represent the corresponding forces expressed in any unit of weight.

When the angle between the forces F and f is known, the magnitude and direction of the resultant R', or resistance R, can be determined by the aid of trigonometry, as follows:

Let u denote the angle in degrees between the two forces F and f, and v = the angle between F and the resultant R', then we have

$$tan.v = \frac{f\ sin.u}{F \pm f\ cos.u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 1.$$

$$\text{Resultant } R' = sec.v(F \pm f\ cos.u). \quad . \quad . \quad . \quad . \quad 2.$$

Use the sign + when u is less than 90°, and − when $u > 90°$.

Example 1. The force $F = 68$ and $f = 42$ pounds, acting at the point c, Fig. 2, at an angle $u = 65°\ 30'$. Required the force and direction of the resultant R' or resistance R.

$$\textit{Formula 1.}\quad tan.v = \frac{42 \times sin.65°\ 30'}{68 + 42 \times cos.65°\ 30'} = 0.44743,$$

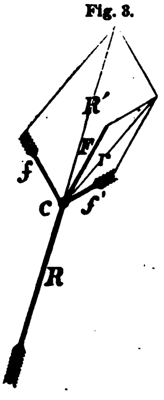
which is the tangent for 24° 6′, the direction of the resultant R' to the force F.

Resultant $R' = \sec.24°\ 6'(68 + 42 \times \cos.65°\ 30') = 93.574$ pounds, the force of resistance required.

"Nystrom's Pocket-Book" contains complete tables of the trigonometrical functions, both natural and logarithmic.

	Natural.	Logarithm.
$sin.65°\ 30'$	= 0.90996	= 9.95902
$sec.24°\ 6'$	= 1.0955	= 10.03961

Fig. 3.



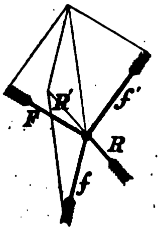
The insertion and use of trigonometrical functions in algebraical formulas can be learned without being well versed in the science of trigonometry.

PROBLEM 3.

When three or more forces are acting in different directions at a point c, Fig. 3, find first the resultant between either two of them, say F and f', which gives the resultant r as a single force. Complete the parallelogram r, f, and the diagonal R' is the resultant of the three forces F, f, f'.

The Formulas 1 and 2 are applied to each two forces or resultants, as in Example 1.

Fig. 4.



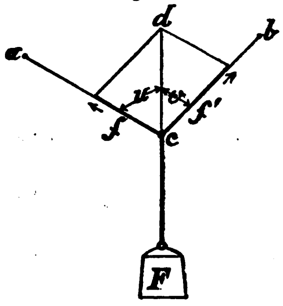
PROBLEM 4.

Forces may be applied so that they nearly counterbalance one another, as in Fig. 4. The resultant R' is smaller than either of the three forces F, f and f'.

PROBLEM 5.

To dissolve a force into two component parts, Fig. 5. Suspend a rope from a to b, and hang a weight F on it at c; then the force F is dissolved into two parts, acting one on the line $c\ a$, and the other on $c\ b$. Draw the line $c\ d$ to represent the magnitude and direction of the force F. Complete the parallelogram, and the side f represents the force acting on $c\ d$, and f' that on $c\ b$.

Fig. 5.



The trigonometrical expression for Fig. 5 will be:

$$F : f' = \sin.(u+v) : \sin.u. \qquad f' = \frac{F \sin.u}{\sin.(u+v)}.$$

$$F : f = \sin.(u+v) : \sin.v. \qquad f = \frac{F \sin.v}{\sin.(u+v)}.$$

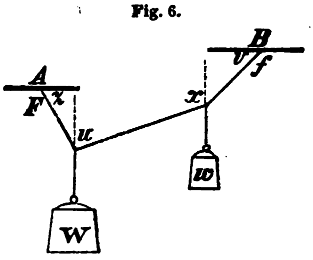
$$f : f' = \sin.v : \sin.u. \qquad f' = \frac{f \sin.u}{\sin.v}.$$

$$F = \frac{f' \sin.(u+v)}{\sin.u} = \frac{f \sin.(u+v)}{\sin.v}.$$

PROBLEM 6.

Let a rope be suspended from two points A and B, and two weights W and w hung upon it. The rope is considered to have no weight, but only stretched by W and w. Let F denote the tension of the rope at A, and f = that at B; then

Fig. 6.



$$f : w = \sin.x : \sin.(x+v.). \qquad f = \frac{w \sin.x}{\sin.(x+v)}.$$

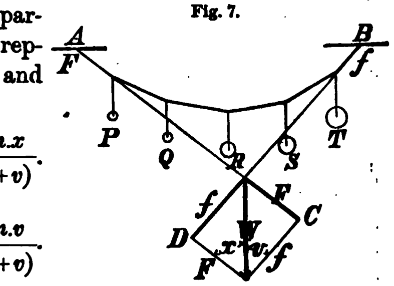
$$F : W = \sin.u : \sin.(u+x) \qquad F = \frac{W \sin.u}{\sin.(u+x)}.$$

PROBLEM 7.

Let a rope be suspended from the points A and B, and any number of weights P, Q, R, S, T hung upon it. Let F denote the force of tension at A, and f = that at B. Draw the lines $A\ C$ and $B\ D$, so that they tangent the rope at A and B. From the point of intersection draw the vertical line W to represent the sum of all the weights

$P, Q, R, S, T = W$. Complete the parallelogram, of which the side F represents the force of tension at A, and the side f that at B.

Fig. 7.



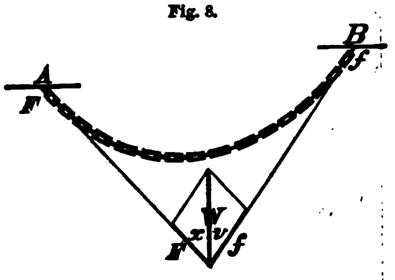
$$F : W = \sin.x : \sin.(x+v). \quad F = \frac{W \sin.x}{\sin.(x+v)}.$$

$$f : w = \sin.v : \sin.(x+v). \quad f = \frac{W \sin.v}{\sin.(x+v)}.$$

PROBLEM 8.

Now let us suppose an infinite number of weights to be suspended on the rope, which is the same as to consider the weight of the rope itself, or that of a chain. Draw the lines and complete the parallelogram as described in the preceding problem. The vertical diagonal W represents the weight of the chain, and the forces F and f are as in the preceding formulas.

Fig. 8.



PROBLEM 9.—THE CATENARY.

The curve formed by a flexible rope or chain suspended from two points is called a *catenary* or *chain-line.*

Let v denote the angle of the curve with the vertical in any point P whose abscissa is x and ordinate y; l = length of the curve $O\,P$. The formulas for the catenary will then be

$$y = \frac{l \sin.^2 v}{\sin.2v} \, hyp.log.cot.\tfrac{1}{2}v.$$

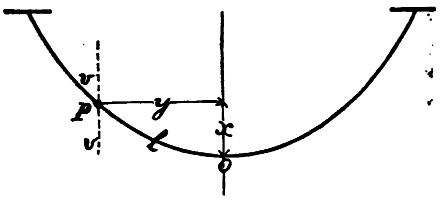
$$x = \frac{l \sin.^2 v}{\sin.2v} \, (cosec.v - 1).$$

$$y = \frac{x \; hyp.log.cot.\frac{1}{2}v}{(cosec.v - 1)}.$$

$$x = \frac{y \, (cosec.v - 1)}{hyp.log.cot.\frac{1}{2}v}.$$

$$l = \frac{x \sin.2v}{\sin.v(1 - \sin.v)}.$$

Fig. 9.



The formulas for the catenary are very difficult to manage, on account of the angle v must be given; but by the aid of the following table the solution becomes very simple:

Table for the Catenary Curve.

Angle v.	Abscissa x.	Ordinate y.	Curve l.	$\frac{y}{x}$.	$\frac{l}{y}$.
30	1.00000	1.31690	1.73210	1.3169	1.3153
40	0.55573	1.01068	1.19175	1.8186	1.1792
45	0.41421	0.88137	1.00000	2.1278	1.1346
50	0.30540	0.76291	0.83910	2.4981	1.1000
54	0.22078	0.65284	0.70021	2.9570	1.0725
60	0.15470	0.54930	0.57735	3.5507	1.0511
62	0.13257	0.50940	0.53171	3.8425	1.0438
64	0.11260	0.47021	0.48773	4.1759	1.0372
66	0.09484	0.43169	0.44523	4.5518	1.0314
68	0.07853	0.39376	0.40403	5.0141	1.0261
70	0.06418	0.35637	0.36397	5.5527	1.0213
71	0.05762	0.33786	0.34433	5.8636	1.0192
72	0.05146	0.31946	0.32492	6.2079	1.0171
73	0.04569	0.30116	0.30573	6.5914	1.0152
74	0.04030	0.28296	0.28675	7.0213	1.0134
75	0.03528	0.26484	0.26795	7.5068	1.0117
76	0.03061	0.24681	0.24933	8.0631	1.0102
77	0.02630	0.22887	0.23087	8.7023	1.0088
78	0.02234	0.21099	0.21256	9.4445	1.0073
79	0.01872	0.19318	0.19438	10.820	1.0062
80	0.01543	0.17542	0.17633	11.372	1.0052
81	0.01247	0.15773	0.15838	12.654	1.0041
82	0.00983	0.14008	0.14054	14.254	1.0033
83	0.00751	0.12248	0.12278	16.309	1.0025
84	0.00551	0.10491	0.10510	19.046	1.0018
85	0.00382	9.08738	0.08749	22.874	1.0013
86	0.00244	0.06987	0.06993	28.613	1.0008
87	0.00137	0.05238	0.05241	38.171	1.0005
88	0.00061	0.03491	0.03492	57.279	1.0002
89	0.00015	0.01745	0.01745	114.586	1.0000

APPLICATION OF THE CATENARY TABLE.

The chain for a suspension bridge of 300 feet span is to hang 60 feet below its supports on the piers. The chain is to support a weight of 52,000 pounds, uniformly distributed in its length. Required the length of the chain and the angle v and strain at the supports? Half the span or $y=150$ feet, for which $x=60$ feet.

$$\frac{y}{x}=\frac{150}{60}=2.5,$$

which corresponds nearly with an angle of $v=50°$ in the table, and the length of half the chain will be $l=150\times 1.1=165$ feet.

The strain at the supports will be

$$F = \frac{W \sin.v}{\sin.2v} = \frac{52000 \times \sin.50^\circ}{\sin.100^\circ} = 40449 \text{ pounds.}$$

The ordinates x and y and the length l for any angle v in the table are found as follows:

When $v = 50^\circ$ at the support, find x and y where $v = 70^\circ$?

$$0.30540 : 0.06418 = 60 : x. \qquad x = \frac{0.06418 \times 60}{0.30540} = 12.609 \text{ feet.}$$

$$0.76291 : 0.35637 = 150 : y. \qquad y = \frac{0.35637 \times 150}{0.76291} = 70.068 \text{ feet.}$$

$$\text{Length } l = 70.068 \times 1.0213 = 71.56 \text{ feet.}$$

The ordinates can thus be calculated for a sufficient number of points in the catenary to define the course of the curve.

The strain at the lowest point, or centre of the catenary, will be

$$w \tan.v = 26000 \times \tan.50^\circ = 30984 \text{ pounds};$$

when v = angle at the piers, and w = half the weight on the whole chain.

The catenary is not a line of the conic sections; its figure has the appearance of a parabola, but is a little fuller at the vertex.

All the curves of the conic sections are of the second order, or of the exponent $n = 2$; whilst the exponent of the catenary is nearly $n = 2.3$.

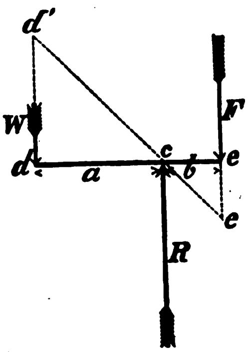
The formula for any parabola is $y = \sqrt[n]{p\,x}$,

when that of the catenary will approach $y = \sqrt[2.3]{p\,x}$.

Length of the curve OP or $l = \frac{1}{3}(2y + \sqrt{y^2 + 9x^2})$.

PROBLEM 10.

Fig. 10.



To find the resultant of two parallel forces F and W, acting at the ends of an inflexible line d, e.

Prolong the line W until $d\,d'$ is equal to F, prolong F until $e\,e'$ is equal to W, then join d' and e', which will cut the inflexible line at c. Draw from c the resultant R, equal and parallel to $W+F$; then R represents the magnitude and direction of the resistance which balances the two forces F and W. The distance $a = d\,c$ is the lever for

the force W, and $b = c\,e$ is the lever for F. which quantities bear the following relation:

$$F : W = a : b, \text{ static momentums } W\,a = F\,b.$$

STATIC MOMENTUM.

The product of a force multiplied by its lever, is called *static momentum*. The resistance at c is called the *fulcrum*. It is supposed in this case that the forces W and F act at right angles to the levers a and b.

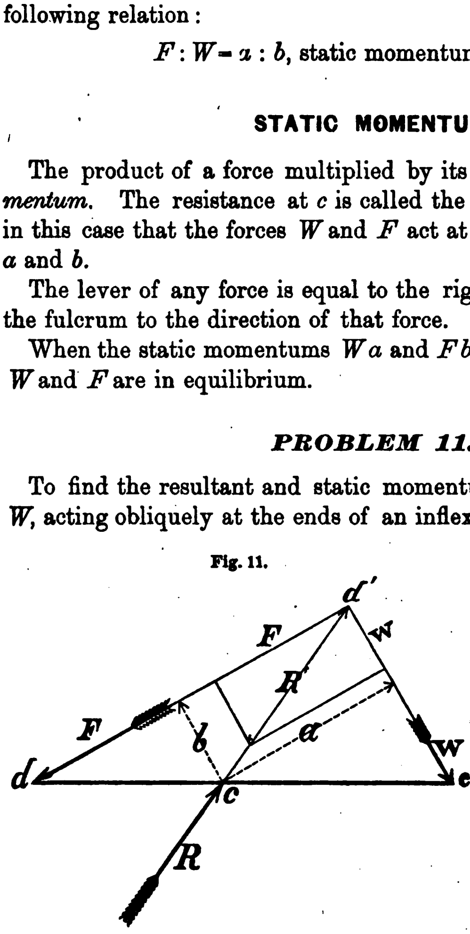
The lever of any force is equal to the right angular distance from the fulcrum to the direction of that force.

When the static momentums $W\,a$ and $F\,b$ are equal, then the forces W and F are in equilibrium.

PROBLEM 11.

To find the resultant and static momentums of two forces, F and W, acting obliquely at the ends of an inflexible line, $d\,e$.

Fig. 11.



Extend the lines of the forces until they meet at d'. Move the forces to d', and complete the parallelogram as shown by Fig. 8; then the diagonal R' is the resultant of the two forces, the continuation of which will cut the inflexible line at the fulcrum c; make $R = R'$, the resistance balancing the two forces. The levers a and b are drawn from the fulcrum c at right angle to the direction of the respective forces.

The analogy of the forces and levers is the same as that in Fig. 10—namely,

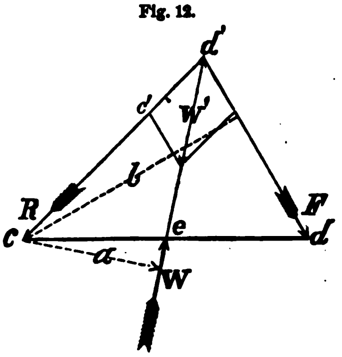
$$F : W = a : b, \text{ static momentums } W\,a = F\,b.$$

The resistance R is less than the sum of the forces $F + W$.

Either one of the points of pressure, d, c or e, may be considered as a fulcrum, as represented in Figs. 12 and 13.

PROBLEM 12.

Fig. 12.



Two forces, F and W, acting against one another on the line $c\,d$, of which c is the fulcrum; to find the magnitude and direction of resistance, in the fulcrum c.

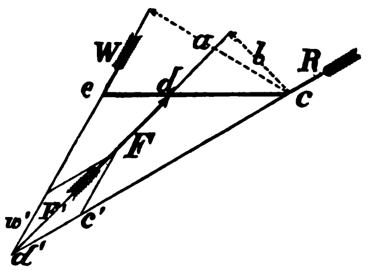
Prolong the forces F and W until they meet at d', from which set off the respective forces F' and W'; complete the parallelogram and prolong $d'\,c'$ until it cut the line $d\,e$ in c, which is the fulcrum for the forces, and $d'\,c' = R$ is the resistance. The levers a and b are drawn from the fulcrum c at right angles to the directions of the respective forces.

$F : W = a : b$, static momentums $Wa = Fb$.

PROBLEM 13.

Two forces, F and W, acting against one another on the line $d\,e$, of which c is the fulcrum, to find the magnitude and direction of resistance, and the fulcrum c.

Fig. 13.



Prolong F and W until they meet at d'. Set off W and F from d', respectively; complete the parallelogram and continue $d'\,c'$ until it cut the line $e\,d$ at c, which will be the fulcrum, and $d'\,c' = R$ is the resistant required.

The levers a and b are drawn from the fulcrum c at right angles to the direction of the respective forces.

$F : W = a : b$, static momentums $Wa = Fb$.

LEVERS.

Levers are classified into three different kinds in reference to the relative position of the force F, weight W, and fulcrum c.

When the fulcrum c is between the force F and weight W, the lever is called the **first kind.** Fig. 14.

When the weight W is between the force F and the fulcrum c, the lever is of the **second kind.** Fig. 15.

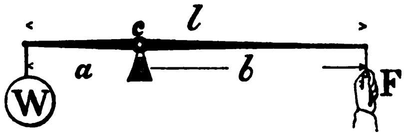
When the force F is between the weight W and the fulcrum c, the lever is of the **third kind.** Fig. 16.

The two forces F and W will be distinguished by considering F the applied force, acting on its lever b, to lift the weight W, acting on its lever a.

PROBLEM 14.

ON THE ELEMENTS OF A LEVER OF THE FIRST KIND.

Fig. 14.



The distance between W and F is denoted by l.

$$F : W = a : b, \quad \text{or} \quad Fb = Wa.$$

$$F = \frac{Wa}{b}, \quad W = \frac{Fb}{a}, \quad b = \frac{Wa}{F}, \quad a = \frac{Fb}{W}.$$

Example 1. Suppose the weight $W = 360$ pounds acting on a lever $a = 2.5$ feet, what force F is required on a lever $b = 6$ feet?

$$F = \frac{Wa}{b} = \frac{360 \times 2.5}{6} = 150 \text{ pounds, the answer.}$$

That is to say, the force $F = 150$ pounds acting on a lever $b = 6$ feet will just balance the weight $W = 360$ pounds on the lever $a = 2.5$ feet, without regard to motion.

Example 2. The force $F = 450$ and the weight $W = 990$ pounds acting on a lever $a = 3.6$ feet, required the lever b, for the force F?

$$b = \frac{Wa}{F} = \frac{990 \times 3.6}{450} = 7.92 \text{ feet, the answer.}$$

When the force F and weight W, and the distance l between them, are given, to find the fulcrum c and the levers a and b.

The lever $a = l - b$, and the lever $b = l - a$.

$$Fb = Wa, \quad \text{or} \quad F(l-a) = Wa. \qquad Fl - Fa = Wa,$$

$$-Fa = Wa - Fl, \quad \text{or} \quad Fa = Fl - Wa.$$

$$Fa + Wa = Fl, \quad a(F+W) = Fl, \quad \text{and} \quad a = \frac{Fl}{F+W}.$$

Example 3. To prove the correctness of the formula, let us assume the same value of the quantities, as in Example 2.

$l = 3.6 + 7.92 = 11.52$ feet. Find the levers a and b.

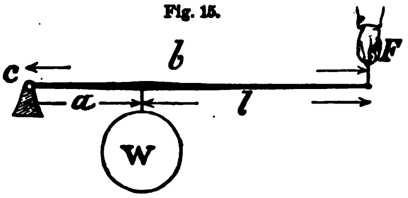
$$a = \frac{Fl}{F+W} = \frac{450 \times 11.52}{450 + 990} = 3.6 \text{ feet, and } b = 11.52 - 3.6 = 7.92 \text{ feet.}$$

PROBLEM 15.

ON THE ELEMENTS OF A LEVER OF THE SECOND KIND.

$$F : W = a : b, \qquad Fb = Wa.$$

Fig. 15.



$$F = \frac{Wa}{b}, \qquad W = \frac{Fb}{a},$$

$$b = \frac{Wa}{F}, \qquad a = \frac{Fb}{W}.$$

Example 4. The weight $W = 3696$ pounds acting on a lever $a = 0.5$ feet; what force F is required on the lever $b = 15$ feet?

$$F = \frac{Wa}{b} = \frac{3696 \times 0.5}{15} = 123.2 \text{ pounds, the answer.}$$

To find the levers and fulcrum when the weight W, force F and distance l are given.

$$b = a + l, \quad a = b - l.$$

$$Fb = Wa, \quad F(a+l) = Wa. \qquad Fa + Fl = Wa,$$

$$Fa - Wa = -Fl, \quad Wa - Fa = Fl.$$

$$a(W - F) = Fl, \quad \text{and} \quad a = \frac{Fl}{W-F}.$$

Example 5. $W = 6396$, $F = 150$ pound, and the distance $l = 9$ feet; required the levers a and b.

$$a = \frac{Fl}{W-F} = \frac{150 \times 9}{6396 - 15} = 0.2115655 \text{ feet} = 2.53877 \text{ inches.}$$

The lever $b = a + l = 9 + 0.2115655 = 9.2115655$ feet.

Example 6. A weight $W=50$ tons is to be lifted by a force $F=0.25$ tons, acting on a lever $b=45$ inches; at what distance shall the weight be applied from the fulcrum c?

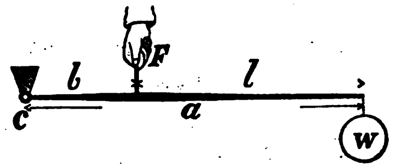
$$a=\frac{Fb}{W}=\frac{0.25\times 45}{50}=0.225 \text{ inches, the answer.}$$

F and W can be expressed in any unit of weight, as well as the levers in any unit of length.

PROBLEM 16.

ON THE ELEMENTS OF A LEVER OF THE THIRD KIND.

Fig. 16.



$F:W=a:b$, or $Fb=Wa$.

$$F=\frac{Wa}{b},\qquad W=\frac{Fb}{a},$$

$$a=\frac{Fb}{W},\qquad b=\frac{Wa}{F}.$$

Example 7. A weight $W=50$ kilograms, acting on a lever $a=4.5$ metres, is to be lifted by a force $F=88$ kilograms; required the lever b?

$$b=\frac{Wa}{F}=\frac{50\times 4.5}{88}=2.557 \text{ metres.}$$

Given the force F, weight W, and the distance l, to find the fulcrum and levers a and b.

$$a=b+l,\qquad b=a-l.$$

$$Fb=Wa, \text{ or } F(a-l)=Wa.$$

$$Fa-Fl=Wa,\qquad Fa-Wa=Fl.$$

$$a(F-W)=Fl, \text{ and } a=\frac{Fl}{F-W}.$$

Example 8. The distance $l=2.3$ yards, $W=5$ hundredweight, and $F=5.5$ hundredweight; required the lever a?

$$a=\frac{Fl}{F-W}=\frac{5.5\times 2.3}{5.5-5}=25.3 \text{ yards, nearly.}$$

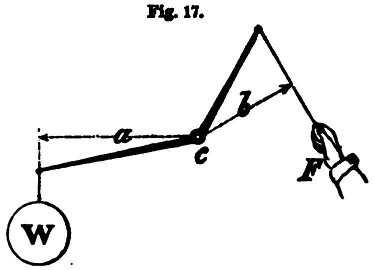
The lever b will be $25.3-2.3=23$ yards.

PROBLEM 17.

We have now considered the levers *a* and *b* to be in a straight line, on which the forces *F* and *W* are acting at right angles; but in practice these elements occur in a great variety of relative positions, of which one is illustrated by Fig. 17.

The apparent lever *a* of the weight inclines under the fulcrum, whilst that of the force *F* rises above. The real or actual levers for these forces are the right-angular distances *a* and *b*; but even these are not in a straight line.

Fig. 17.

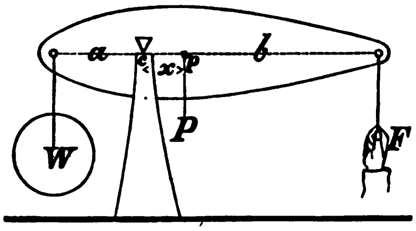


Whatever may be the form of the apparent levers, or whatever may be the direction of the forces, the actual levers must be considered from the fulcrum, at right angles to the directions of the respective forces. The static momentums of equilibrium and the formulas for the elements are precisely the same as that explained for the three kinds of simple levers.

PROBLEM 18.

We have heretofore considered the levers to be inflexible lines without weight, which will answer when the centre of gravity of the material levers is in the fulcrum, like that of a weighing balance or that of a wheel; but this centre of gravity is often located at a considerable distance from the fulcrum, as may be illustrated by Fig. 18, which is a lever of the first kind. The levers *a* and *b* are in the form of a beam, resting on the fulcrum *c*, and its centre of gravity at *p*. Let the weights of the beam be represented by *P*, acting on the lever *x*; there will be two static momentums, *F b* and *P x*, on one side of the fulcrum, against one, *W a*, on the other side.

Fig. 18.



$$Wa = Fb + Px, \qquad Fb = Wa - Px, \qquad Px = Wa - Fb.$$

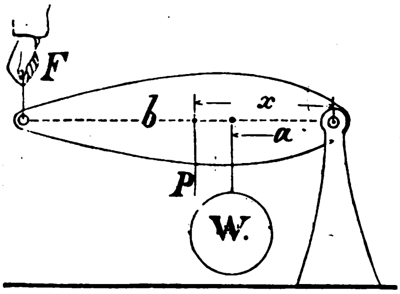
$$W = \frac{Fb+Px}{a}, \qquad F = \frac{Wa-Px}{b}, \qquad P = \frac{Wa-Fb}{x}.$$

$$a = \frac{Fb+Pa}{W}, \qquad b = \frac{Wa-Px}{F}, \qquad x = \frac{Wa-Fb}{P}.$$

The centre of gravity p may be found experimentally by balancing the beam over a sharp edge, when the distance x can be measured from the fulcrum c. It is here supposed that the levers a and b are in a straight and horizontal line.

The form of the beam affects the location of the centre of gravity p, but when this centre is known the shape of the beam does not affect the static momentums.

Fig. 19.



The pressure on the fulcrum c is equal to $W+P-F$.

PROBLEM 19.

This is a lever of the second kind, in which all the static momentums are on one side of the fulcrum.

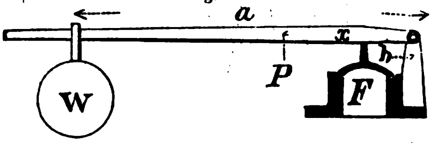
$$Fb = Wa + Px, \qquad Wa = Fb - Px, \qquad Px = Fb - Wa.$$

$$F = \frac{Wa+Px}{b}, \qquad W = \frac{Fb-Px}{a}, \qquad P = \frac{Fb-Wa}{x}.$$

$$b = \frac{Wa+Px}{F}, \qquad a = \frac{Fb-Px}{W}, \qquad x = \frac{Fb-Wa}{P}.$$

PROBLEM 20.

Fig. 20.



This is a lever of the third kind, representing a safety-valve for a steam-boiler.

$$Fb = Wa + Px, \qquad Wa = Fb - Px, \qquad Px = Fb - Wa.$$

$$F = \frac{Wa+Px}{b}, \qquad W = \frac{Fb-Px}{a}, \qquad P = \frac{Fb-Wa}{x}.$$

$$b=\frac{Wa+Px}{F},\qquad a=\frac{Fb-Px}{W},\qquad x=\frac{Fb-Wa}{P}.$$

The formulas are the same as those for the lever of the second kind, Fig. 19.

The centre of gravity of the lever is found by balancing it over a sharp edge, and the distance x is measured from the fulcrum. The weight of the lever is found by weighing it, and thus the momentum Px is obtained simply by multiplying the weight P by the distance x, which is a constant quantity in the graduation of the lever for different pressures of steam.

Example 9. Suppose the safety-valve to be three inches in diameter, and the lever to be graduated between pressures of 20 and 50 pounds to the square inch. Area of the valve is 7 square inches, and $7\times 50=350$ pounds, which is the force F. The lever is found to weigh $P=10$ pounds and $x=18$ inches, which momentum $Px=10\times 18=180$. The lever $b=4$ inches and $a=48$ inches where the weight W is expected to indicate a steam pressure of 50 pounds to the square inch. Then the weight will be

$$W=\frac{Fb-Px}{a}=\frac{350\times 4-10\times 18}{48}=27.08 \text{ pounds.}$$

Now find the lever a, or at what distance from the fulcrum shall the weight be placed for balancing a steam pressure of 20 pounds to the square inch. $F=20\times 7=140$ pounds, and the lever

$$a=\frac{Fb-Px}{W}=\frac{140\times 4-10\times 18}{27.08}=14.03 \text{ inches.}$$

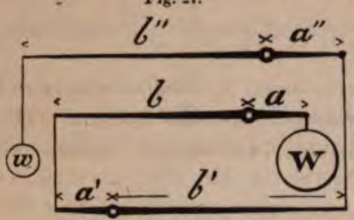
Measure off this distance from the fulcrum and mark it with the number 20. Divide the distance between the first and second positions of the weight into $50-20=30$ equal parts, and number them from 20 to 50. The lever is thus graduated for the required pressures of steam.

For greater accuracy it is necessary to weigh the valve with the lever for the weight P, and it is also necessary to place the valve on the lever at its proper distance from the fulcrum, when the lever is balanced over a sharp edge, for finding its centre of gravity.

PROBLEM 21.

ON THE COMBINATION OF LEVERS, AS REPRESENTED BY Fig. 21.

Fig. 21.



Levers can be combined in a great variety of ways, but are generally arranged so that a short lever acts on a long one.

$$W : w = b\,b'\,b'' : a\,a'\,a'', \quad \text{or } W\,a\,a'\,a'' = w\,b\,b'\,b''.$$

$$W = \frac{w\,b\,b'\,b''}{a\,a'\,a''}, \quad \text{and} \quad w = \frac{W\,a\,a'\,a''}{b\,b'\,b''}.$$

That is to say, the big weight W is to the small weight w as the product of all the long levers b is to the product of all the short levers a.

The big weight multiplied by the product of the short levers is equal to the small weight multiplied by the product of the long levers.

This rule will hold good for all combinations of levers, but without considering the weight and centre of gravity of the levers. The static momentum of the weights of the levers is determined by hanging small weights at W until it balances the levers without any weight at w.

PROBLEM 22.—PLATFORM SCALE.

This figure represents a platform scale. $A\ B$ is the platform, resting on levers underneath.

Fig. 22.

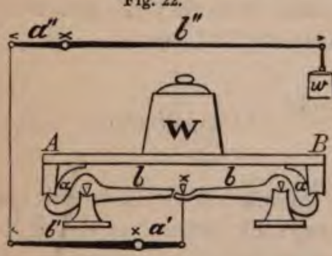
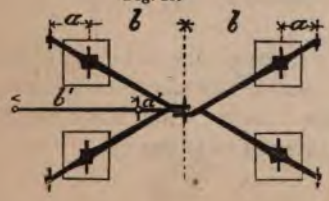


Fig. 23.



The lower figure represents the plan of the levers, with the platform removed.

The platform rests on four points, from which the levers extend diagonally to the centre, and there are also four fulcrums, all of which only serve to give parallel motion to the platform, upon which the weight W can be placed at one corner, or be irregularly distributed, without effecting any difference in the weight w. Although the platform and weight W rest upon four diagonal levers, the real levers of the static momentums must be considered parallel to the

sides of the platform, as marked on the figures, and only one of them enters into the formulas and calculations.

$$W : w = b\, b'\, b'' : a\, a'\, a'', \quad \text{or} \quad W a\, a'\, a'' = w\, b\, b'\, b''.$$

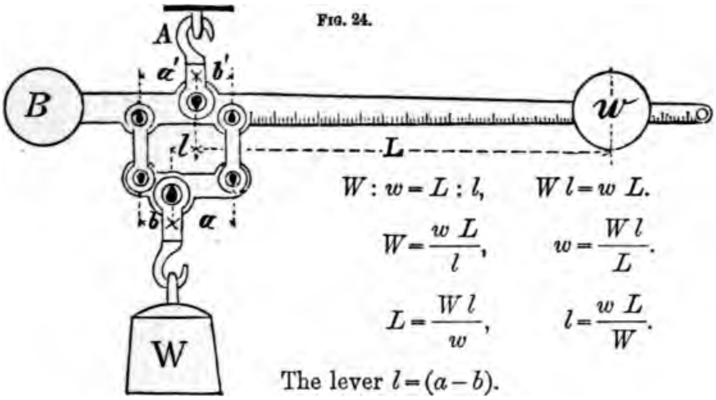
$$W = \frac{w\, b\, b'\, b''}{a\, a'\, a''}, \quad \text{and} \quad w = \frac{W a\, a'\, a''}{b\, b'\, b''}.$$

The platform and levers ought to be so arranged and proportioned that they balance one another, or that they are in equilibrium without the weight W and w.

DIFFERENTIAL BALANCE.

Fig. 24 represents a convenient balance-scale for weighing heavy weights. It is much used in iron-foundries, where the balance with the weight is hoisted in a crane for weighing.

Fig. 24.



$$W : w = L : l, \qquad W l = w L.$$

$$W = \frac{w L}{l}, \qquad w = \frac{W l}{L}.$$

$$L = \frac{W l}{w}, \qquad l = \frac{w L}{W}.$$

The lever $l = (a - b)$.

The object of the links and the short lever is to bring the weight close to the fulcrum, or, more correctly, to obtain a short lever l.

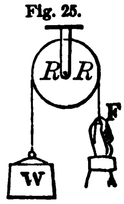
There is not room enough on the main-balance to bring the direction of the action of the weight W sufficiently close to the fulcrum for weighing heavy weights. The levers a and b ought to be equal to a' and b', respectively. $a + b = a' + b'$, and $a - b = a' - b'$.

The difference between a and b is generally not made so great as shown in the illustration. For a lever $L = 8$ feet, the lever l is only a fraction of an inch, and can be made as small as desired by making $(a - b)$ small. The scale should be well balanced by the ball B, without the weights W and w.

PULLEYS.

PROBLEM 23.—FIXED PULLEYS.

Fig. 25.

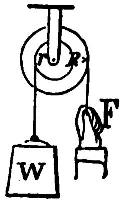


Let a force F be applied on a rope Fig. 25, extending over a fixed pulley R R to lift a weight W; then $F : W = R : R$, or $F = W$.

The radii of the pulley act as levers for the forces, and as the radii are alike on both sides, the force F will be equal to the weight W.

Should the forces F and W act on different radii, Fig. 26, then,

Fig. 26.



$$F : W = r : R, \qquad F R = W r.$$

$$F = \frac{W r}{R}, \qquad W = \frac{F R}{r}.$$

$$R = \frac{W r}{F}, \qquad r = \frac{F R}{W}.$$

PROBLEM 24.

DIFFERENTIAL PULLEY-BLOCKS.

Fig. 27.

The endless rope or chain, $f f$, passes over a pulley with two grooves of different radii R and r, as represented by Fig. 27. The weight W is equally divided on the ropes or chains $f f$, which tension will consequently be $2f = W$, or $f = \frac{1}{2} W$.

The momentums on each side of the centre will be

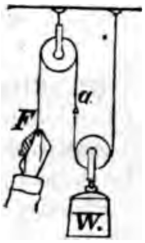
$$f R = f r + F R, \quad \text{or} \quad F R = f R - f r.$$

$$F = \frac{f(R-r)}{R}, \text{ but } W = 2f \text{ and } F = \frac{W(R-r)}{2R}, \quad W = \frac{2 F R}{R-r}.$$

Example. The weight $W = 1500$ pounds, the radius of the large sheave is $R = 6$ inches, and the small sheave $r = 5$ inches. Required the force F?

$$F = \frac{1500(6-5)}{2 \times 6} = 125 \text{ pounds.}$$

Fig. 28.



PROBLEM 25.—MOVABLE PULLEYS.

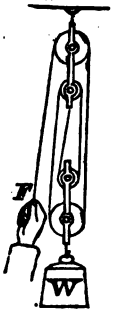
Single movable pulley, Fig. 28,

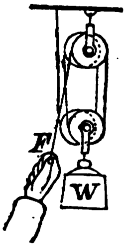
$$F : W = 1 : 2.$$

$$F = \tfrac{1}{2} W, \qquad W = 2F.$$

If the force is applied at a and acts upward, the result will be the same.

Fig. 29. Fig. 30.





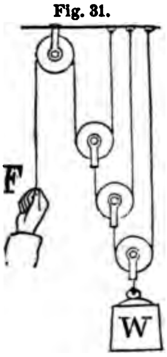
Double movable pulleys, Figs. 29 and 30,

$$F : W = 1 : 4, \qquad F = \tfrac{1}{4} W, \qquad W = 4F.$$

The force F pulls on only one rope, whilst the weight W hangs on four ropes; for which the weight is four times the force. For three, four, etc. movable pulleys there will be six, eight, etc. ropes, and the weight will be so much greater than the force.

PROBLEM 26.—COMPOUND PULLEYS.

Fig. 31.



Let n denote the number of movable pulleys, then

$$F : W = 1 : 2^n, \qquad W = F2^n.$$

$$F = \frac{W}{2^n}.$$

Example. What weight W can be lifted by $n = 3$, or four movable pulleys, the force $F = 300$ pounds? $W = F2^n = 300 \times 2^3 = 300 \times 8 = 2400$ pounds, the answer.

Fig. 31 represents three, or $n = 3$ movable pulleys.

MOTION OF THE FORCE AND WEIGHT.

Let M denote the motion of the force F, and m = motion of the weight W; then

$$M : m = W : F, \qquad FM = Wm.$$

$$M = \frac{Wm}{F}, \qquad m = \frac{FM}{W}.$$

This proportion will hold good in all kinds of levers and pulleys. That is to say, *the motion of the force is to the motion of the weight as the weight is to the force.*

STABILITY AND EQUILIBRIUM.

Stability of a body is that state of rest which cannot be disturbed by an infinitely small force.

Equilibrium of bodies is that state of rest which can be disturbed by an infinitely small force.

Stability and equilibrium are generally referred to the force of gravity acting vertically or at right angles to the surface of the earth.

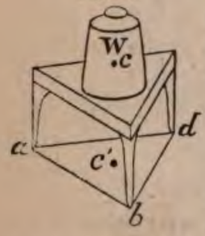
The force with which a body rests on a foundation is equal to the weight of the body acting in the direction of the vertical line passing through its centre of gravity. Should the body rest on only one point in the aforesaid line, it will be in equilibrium, and an infinitely small force can upset it; the body has therefore no stability.

Let the body rest on two points in a vertical plane passing through its centre of gravity, and the two points of support are one on each side of the vertical from the same centre of gravity; then the body will be in equilibrium in the direction at right angles to the plane, and stable in the direction of the plane; the body must therefore rest on more than two points to be absolutely stable.

PROBLEM 27.

Let a body W be supported on a tripod or three points a, b, d in a horizontal plane. Draw from the centre of gravity of the system the vertical line c, c', which must fall inside of the triangle a, b, d to make the weight and tripod stable, and an appreciable force is required to upset the system.

Fig. 32.



The term system means the tripod and the weight, or any number of bodies rigid together and supported at the points a, b, c. The system can be upset in any direction, but it is generally considered to be upset in the direction over either one of the sides in the base-triangle, which is called the *line of fulcrum*. The static lever upon which the system acts is the rectangular distance from c' to the line of fulcrum.

The static momentum of the force must be equal to the momentum of stability to bring the force and weight in equilibrium, and any additional force, however small, will upset the system.

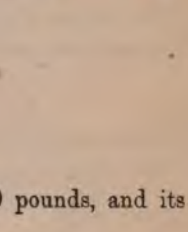
MOMENTUM OF STABILITY.

The weight of the system, multiplied by its static lever, is called the *momentum of stability.*

The height of the centre of gravity c does not affect the momentum of stability. The force required to upset the system depends upon its lever of action. The lever of the force is the rectangular or shortest distance between the direction of the force and the line of fulcrum.

PROBLEM 28.

Fig. 33.



A body W, with its centre of gravity at c, is resting on the plane $A\ B$. A force F is applied to upset the body over the fulcrum at A. Continue the direction of the force, and draw the line b at right angles to $F f$ from the fulcrum, then b is the lever for the force, and a is the lever for the weight of the body.

$$F : W = a : b, \qquad F b = W a.$$

$$F = \frac{W a}{b}, \qquad W = \frac{F b}{a}.$$

$$b = \frac{W a}{F}, \qquad a = \frac{F b}{W}.$$

Example. The weight of the body is $W = 3600$ pounds, and its lever $a = 1.8$ feet.

The direction of the force F is such that its lever $b = 5$ feet. Required the force F?

Fig. 34.

$$F = \frac{W a}{b} = \frac{3600 \times 1.8}{5} = 1296 \text{ pounds.}$$

Let the same body be upset over the fulcrum at B, the lever $a = 4$ feet; the force F being applied so that the lever $b = 7$ feet. Required the force F?

$$F = \frac{3600 \times 4}{7} = 2059 \text{ pounds.}$$

PROBLEM 29.

RETAINING WALLS FOR WATER.

Fig. 35 represents a wall of hydraulic masonry erected for retaining water, which presses on the inside with a tendency to upset the wall over the fulcrum at A.

Fgi. 35.

For simplicity in the formulas and calculations we will assume a definite length of the wall, say one foot. The pressure of the water against the wall increases with the depth d, so that if the line $n\ o$ represents the water-pressure per unit of surface at the bottom of the water, then any other horizontal line in the triangle m, n, o represents the corresponding pressure at that depth, and the area of the triangle represents the total pressure against the wall. The centre of pressure must therefore be at the same height as that of the centre of gravity of the triangle, which is at $\frac{1}{3}\,d$ from the bottom, where the resultant of all the forces acts on the lever b to upset the wall.

The weight of a cubic foot of water at 60° Fahr. is 62.33 pounds, and the total pressure, or the force F, acting on the side of the wall, will be

$$F = \tfrac{1}{2} \times 62.33\,d = 31.16\,d.$$

The lever of this force is $b = \frac{1}{3}\,d$.

The static momentum of the force will be

$$F\,b = \tfrac{1}{3}\,F\,d, \qquad \text{but } F = 31.16\,d,$$

or the static momentum $= \frac{1}{3} \times 31.16\,d^2 = 10.3866\,d^2$.

The weight per cubic foot of hydraulic masonry may be assumed as follows:

Brick, 90 pounds,
Granite, 140 pounds.

Let h denote the height of the wall, t = its thickness in feet, and w = weight per cubic foot of the materials in the wall.

Then, the weight of the wall will be

$$W = h\,t\,w,$$

when the momentum of stability will be, without water-pressure,

$$W\,a = h\,t\,w\,a.$$

From this momentum of stability must be subtracted the static momentum of the water-pressure, and the remainder will be the real momentum of stability of the wall with water-pressure.

Let W' denote the weight of the real momentum of stability, or that part of the weight of the wall which keeps it in proper position against the water-pressure; then

$$W'a = h\,t\,w\,a - 10.3866d^2.$$

Example. Suppose the wall to be of brick masonry, for which $w = 90$, the height $h = 12$ feet, thickness $t = 2$ feet, and the depth of water $d = 10$ feet. Required the real momentum of stability? The lever $a = 1$ foot. $10.3866 \times 10^2 = 1038.66$ foot-pounds, the static momentum of the water.

$$12 \times 2 \times 90 \times 1 = 2160 \text{ foot-pounds.}$$

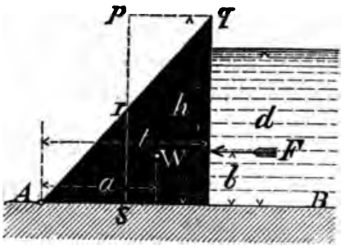
Real momentum of stability $= 2160 - 1038 = 1122$ foot-pounds.

That is to say, the real static momentum of the wall is slightly more than the static momentum of the water-pressure, or that the momentum of stability of the wall without water-pressure is more than double the static momentum of the water.

This rule will hold good for any length of wall when the dimensions are in feet and the pressures in pounds. In practice, the momentum of stability of the wall alone should be at least four times the static momentum of the water-pressure.

PROBLEM 30.

Fig. 36.



Let the materials in the wall in the preceding figure and example be formed into a wall of this figure. That is to say, the materials in the triangle p, q, r are placed at the base s, A, r, and formed into a solid wall A, B, q.

The static momentum of the water-pressure, and also the weight of the wall W will be the same as in the preceding example, but the base A, B will here be doubled, or $t = 4$ feet, instead of two. The lever of the weight W will be $a = \frac{2}{3} \times 4 = 2.66$ feet, instead of one foot, and the momentum of stability will be

$$2160 \times 2.66 = 5745.6 \text{ foot-pounds.}$$

The real momentum of stab. = 5745.6 − 1038.66 = 4707 foot-pounds, or over four times that in the preceding example, where the wall of the same amount of materials was vertical on the outside.

Now let the same wall, with the same materials and dimensions, be turned so that the water-pressure falls on the inclined side, and the fulcrum of the wall at B.

The lever of the weight W will now be only $\frac{1}{3} \times 4 = 1.33$ feet, and the momentum of stability without water-pressure will be

$$2160 \times 1.33 = 2872.8 \text{ foot-pounds.}$$

PROBLEM 31.

Fig. 37.

The water-pressures act at right angles on the inside of the wall A, C, the centre of which will be at $\frac{1}{3}$ of the depth d, or at $\frac{1}{3}$ of A, C from A. Continue the force F, and draw b at right angles to it from the fulcrum B. With the assumed dimensions of the wall, 12 feet high and 4 feet base, and depth of water 10 feet, the lever b will be 1.2648, $a = 1.333$ and the side A, $C = 10.54$ feet.

The force $F = 10.54 \times 62.33 \times 0.5 = 3284.79$ pounds.

Static moment. $= Fb = 3284.79 \times 1.2648 = 4154.6$ foot-pounds.

Real moment. stab. $= 4154.6 - 2872.8 = 1281.8$ foot-pounds.

This proves that a wall of the assumed dimensions has greater stability when the water presses on the vertical side than when on the inclined side.

The retaining wall and its inclined side may be so proportioned that the static momentum of the water-pressure increases the stability of the wall, as in Fig. 38.

PROBLEM 32.

ELEMENTS OF THE RETAINING WALL.

Fig. 38.

$$tan.v = \frac{h}{t}. \qquad t = \frac{h}{tan.v}. \qquad h = t\, tan.v.$$

$$a = \tfrac{1}{3}\, t. \qquad W = \tfrac{1}{2}\, h\, t\, w.$$

Moment. stab. $= Wa = \frac{1}{2}\, h\, t\, w \times \frac{1}{3}\, t = \frac{1}{6}\, h\, t^2 w.$

ELEMENTS OF THE WATER-PRESSURE.

The inclined side $A, C = d\ cosec.v.$

The force $F = \frac{1}{2}\ d\ cosec.v.$

The distance δ from A to the centre of pressure will be $\delta = \frac{1}{3}\ d$ $cosec.v.$

$$b : \delta = \mp t \pm \delta\ sec.v : \delta\ sec.v.$$

$$b = \frac{\delta\,(\mp t \pm \delta\ sec.v)}{\delta\ sec.v} = \frac{\mp t}{sec.v} \pm \delta = \mp t\ cos.v \pm \tfrac{1}{3}\ cosec.v.$$

$$b = \mp t\ cos.v \pm \frac{d}{3\ sin.v}.$$

Subtract the smallest term from the largest, and the remainder is the lever b.

When $t\ cos.v > \frac{d}{3\ sin.v}$, the direction of the force F passes outside of the fulcrum B, as in Fig. 37.

When $t\ cos.v < \frac{d}{3\ sin.v}$, the direction of the force F passes inside of fulcrum B, as in Fig. 38, and the static momentum $F\,b$ increases the stability of the wall.

The trigonometrical functions can be expressed by the dimensions of the wall, as follows:

R = length of the hypotenuse of the wall.

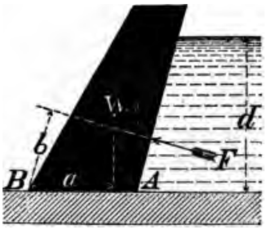
$$sin.v = \frac{h}{R}, \quad cos.v = \frac{t}{R}, \quad sec.v = \frac{R}{t}, \quad cosec.v = \frac{R}{h}.$$

$$F = \frac{d\,R}{2\,h}, \quad \delta = \frac{d\,R}{3\,h}.$$

$$b = \mp \frac{t^2}{R} \pm \frac{d\,R}{3\,h}.$$

PROBLEM 33.

Fig. 39.



Whatever may be the shape and position of a retaining wall, its momentum of stability is $W\,a$, and the static pressure of the water acts at right angles to the surface of the wall.

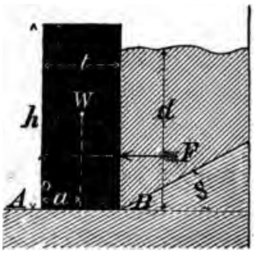
The force of the water-pressure in pounds is equal to the area of the pressed surface in square feet, multiplied by half the depth in feet × 62.33.

The centre of pressure is at one-third of the depth from the bottom.

PROBLEM 34.

RETAINING WALLS FOR EARTHWORK.

Fig. 40.



The action of earth or other granular substances, like sand, gravel, grain, etc., on retaining walls is the same as that described for water, and the momentums are calculated by the same formulas, with the only exception that the natural slope of the granular materials diminishes the force F as the cosine for that slope.

The natural slope of a granular substance is the greatest angle with the horizon at which it will repose in a heap. Let s denote the angle of natural slope and W = weight per cubic foot of the material retained by the wall; which values for some substances are contained in the following table:

Natural Slope and Weight of Granular Substances.

Granular Substances Loosely Heaped.	Slope. s.	$\cos.s$.	Weight, w.
Lime (powder)	45	0.70711	
Saw-dust, wheat-flour	44	0.71934	
Broken stone or coal	43	0.73135	
Malt-flour	40	0.76604	
Sand (moist)	39	0.77715	95
Sand (dry)	38	0.78801	94
Malt-corn	37	0.79863	47
Wheat, rye and corn	36	0.80902	45
Peas	35	0.81915	48
Gravel and earth	35 to 40	0.8 to 0.75	80 to 100

The force F per foot of length of the wall, Fig. 40, will be

$$F = \tfrac{1}{2} w\, d \cos.s.$$

For safe calculation we may assume $w = 100$ pounds per cubic foot of earth or gravel pressing against the wall, and $s = 32°\ 51'$ the safety angle of natural slope, which cosine is 0.84; then the force F will be

$$F = \tfrac{1}{2} \times 100 \times 0.84\, d = 42\, d.$$

The static momentum of the force F will be as described for Fig. 35—namely,

$$F b = \tfrac{1}{3} F d, \qquad \text{but } F = 42\, d.$$

$$F b = \tfrac{1}{3} \times 42\, d^2 = 14\, d^2.$$

The momentum of stability of the wall alone will be

$$W a = h\, t\, w\, a,$$

of which the weight $w = h\, t\, w$, and the lever $a = \frac{1}{2} t$, when the sides of the wall are vertical.

$$W a = \tfrac{1}{2} h\, w\, t^2,$$

w = weight per cubic foot of the materials in the wall, which is 90 for brick, 120 for rubble concrete and 140 pounds for granite.

Subtract the static momentum of the earth-pressure from the momentum of stability of the wall alone, and the remainder will be the real stability of the wall.

For safety in practice the momentum of stability of the wall alone ought to be at least four times the static momentum of the earth-pressure, or

$$\tfrac{1}{2} h\, w\, t^2 = 4 \times 14\, d^2, \quad \text{or} \quad h\, w\, t^2 = 112\, d^2.$$

$$h = \frac{112\, d^2}{w\, t^2}, \qquad \text{and } t = \frac{10.583\, d}{\sqrt{h\, w}}.$$

When the height h of the retaining wall is equal to the depth d of the earth-pressure, we have

$$w\, t^2 = 112\, d, \text{ of which } t = \sqrt{\frac{112\, d}{w}}.$$

Example. The depth of earth-pressure is $d = 20$ feet, to be retained by a vertical granite wall of $w = 140$ pounds to the cubic foot. Required the thickness t of the wall.

$$t = \sqrt{\frac{112 \times 20}{140}} = 4 \text{ feet.}$$

This example is for a wall of cut granite block and of first-class workmanship.

The foundation of the wall should be much thicker, depending on the softness of the ground upon which it stands.

A brick wall for retaining earth, of 120 feet high, should be 5 feet thick.

PROBLEM 35.

Fig. 41.

Walls are often built for retaining earth of greater height than that of the wall, as illustrated by the accompanying figure, in which case the centre of pressure of the earth will not be at one-third from the bottom, as in the former cases.

Let e denote the height of centre of pressure above the bottom of the wall; then

$$e=\frac{2d}{3}-\sqrt{4(d-h)^2+d^2},$$

when the earth above the wall rises with its natural slope to the height d.

Draw the force F at right angles to B 2 through the centre of pressure, and continue it past the fulcrum at A, as shown by the dotted line, and the static momentum of the force of earth-pressure can thus be found graphically on the drawing.

The mean height of the column of earth pressing against the wall is $d-\frac{1}{2}h$, and the area of the base of that column is the line $B\,2\times 1$ foot, and the force $F=(B\,2)\,w\,(d-\frac{1}{2}h)$.

The centre of gravity of an irregular four-sided section of the wall is found as follows:

Divide the base A, B and the top 1, 2 each into two equal parts, and join the middle points 7 and 8 with the opposite corners, as shown by the figure 41; divide each of these four lines into three equal parts, join 3, 4 and 5, 6, and the intersection at W will be the centre of gravity of the four-sided section of the wall, and the lever a is thus found.

The area of the four-sided section is equal to that of the two triangles A, B, 7 and 1, 2, 8, which, multiplied by the weight w per cubic foot of the material in the wall, will be the weight W.

In practice the momentum of stability of the wall alone should be at least four times the static momentum of earth-pressure, or

$$W a = 4\,F b.$$

PROBLEM 36.

ON THE STABILITY OF TOWERS TO THE FORCE OF WIND.

The momentum of stability of a tower or any other structure exposed to the wind should be greater than the static momentum of the greatest storm to which the object may be exposed. For safety in practice, the stability should be at least four times the momentum of the wind.

The static momentum is equal to the force of the wind acting on the object, multiplied by the height of the centre of gravity of the surface acted upon.

The following table shows the force of wind in pounds per square foot at different velocities:

Table of Velocity and Force of Wind.

Miles per hour.	Feet per second.	Force per sq. ft.-pound.	Common Appellation of the Force of Wind.
1	1.47	0.005	Hardly perceptible.
2	2.93	0.020	Just perceptible.
3	4.4	0.044	
4	5.87	0.079	Gentle pleasant wind.
5	7.33	0.123	
6	8.8	0.177	
7	10.25	0.241	
8	11.75	0.315	Pleasant brisk gale.
9	13.2	0.400	
10	14.67	0.492	
12	17.6	0.708	
14	20.5	0.964	
15	22.00	1.107	
16	23.45	1.25	
18	26.4	1.55	Very brisk.
20	29.34	1.968	
25	36.67	3.075	
30	44.01	4.429	High wind.
35	51.34	6.027	
40	58.68	7.873	
45	66.01	9.963	Very high.
50	73.35	12.30	
55	80.7	14.9	
60	88.02	17.71	Storm or tempest.
65	95.4	20.85	
70	102.5	24.1	Great storm.
75	110.	27.7	Hurricane.
80	117.36	31.49	
100	146.66	50.	Tornado, tearing up trees, etc.

The highest known wind is termed "tornado," which moves with a velocity of 100 miles per hour, or 146 feet per second, and exerts a

pressure of 50 pounds per square foot on a surface stationed at right angles to the direction of the wind.

Let A denote the area in square feet of the structure exposed to the wind, the greatest force of which will be 50 A; b = height of the centre of gravity of the exposed surface above the ground. The greatest static momentum of the wind will then be 50 $A\,b$, and the minimum stability of the structure should be limited to

$$4 \times 50\,A\,b = 200\,A\,b.$$

Fig. 42.

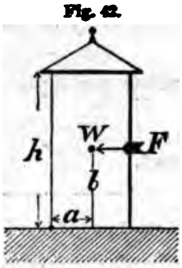


Fig. 42 represents a square tower with parallel sides exposed to a tornado. The momentums will then be

Stability, $Wa = Fb$ static momentum.

Practically, the stability Wa ought to be four times the static momentum Fb.

$$Wa = 4\,Fb, \text{ or } Wa = 200\,A\,b,$$

$$\text{and } W = \frac{200 A\,b}{a},$$

in which W = the whole weight of the tower in pounds, A = area of one side of the tower facing the wind.

Example. Suppose the tower to be four feet square and 20 feet high, from which $a = 2$ and $b = 10$ feet. The area of the side will be $A = 4 \times 20 = 80$ square feet. What weight of the tower is required to maintain it stable to a tornado?

$$W = \frac{200A\,b}{a} = \frac{200 \times 80 \times 10}{2} = 80000 \text{ pounds.}$$

It is supposed that the wind acts at right angles on one side of the tower, but if acting in the direction of the diagonal of the square section, a greater surface will be exposed, but at such angle to the wind that the acting force will be the same as when blowing directly on only one side.

On a round tower of diameter equal to the side of the square the force of the wind is only one-half of that on the square tower.

PROBLEM 37.

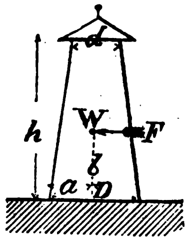
Fig. 43 represents a tower or chimney of the form of a conic frustum of diameters d at the top and D at the base, h = height of the tower; all in feet.

The height of the centre of gravity b is calculated from the following formula:

$$b = \frac{h}{2} - \frac{h}{6}\left(\frac{D-d}{D+d}\right).$$

Example. The height of the tower $h = 260$ feet, diameter at the top $d = 10$ feet, and $D = 25$ feet at the base. Required the height of the centre of gravity of the surface of the tower?

Fig. 43.



$$b = \frac{260}{2} - \frac{260}{6}\left(\frac{25-10}{25+10}\right) = 111.43 \text{ feet.}$$

The projecting area to the wind is

$$\frac{h}{2}(D+d) = \frac{260}{2}(25+10) = 4550 \text{ square feet.}$$

Of this area only one-half is effective to the force of the wind, or $A = 2275$ square feet. The static momentum of a tornado will then be

$$F\,b = 50\,A\,b = 50 \times 2275 \times 111.43 = 12675162.5 \text{ foot-pounds.}$$

The practical momentum of stability of the tower should be at least four times this static momentum.

If the materials in the tower and that in the base it stands on were hard enough to stand the crushing force at the fulcrum in upsetting the tower, the lever of the momentum of stability would be half the diameter D of the base; but as such is not the case, in practice a deduction of that lever must be made in accordance with the softness of the materials acted upon at the fulcrum.

When the base of the tower is square, and the force of the wind acts at right angles to one of its sides, the fulcrum will be a line; whilst on a circular base the fulcrum will be a point in which the whole weight of the tower acts to crush the materials.

In ordinary good brickwork the lever of the momentum of stability may be taken at 0.7 of the radius of the circular base.

The weight of the tower in the preceding example should then be

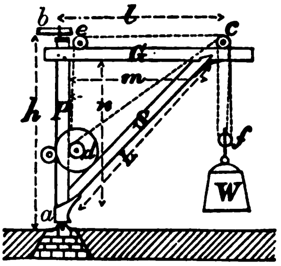
$$W = \frac{4Fb}{a} = \frac{4 \times 12675162.5}{0.7 \times 12.5} = 5794368 \text{ pounds.}$$

A small deduction ought also to be made of the lever in a square base, where it may be taken at 0.9 of half the side of the square.

The cohesive force of the materials increases the stability of the structure when the masonry is perfectly solid at the base.

CRANES.

Fig. 44.



The ordinary crane for hoisting purposes is represented by Fig. 44.

The crane is held in position or supported by the shoe a and cap b. The dotted lines represent the chain or rope by which the weight W is raised.

The chain may run either direct from the barrel d to the blocks c and f, or from d via e, c to f.

The body of the crane is composed of the post P, gib G and stay S.

The weight W acts on the lever l, and the reaction of force F in the supports a and b, acts on the lever h.

Static momentums $F\,h = W\,l$.

$$F = \frac{W\,l}{h}, \qquad W = \frac{F\,h}{l}, \qquad h = \frac{W\,l}{F}, \qquad l = \frac{F\,h}{W}.$$

These are the principal elements of the crane.

When the stay S is in the direction a, c, the force of compression S of the stay will be, when L = length of the stay,

$$S = \frac{W\,L}{h}.$$

Let m denote the distance from the post P to where the stay S bears the gib G, and n = distance from the gib G to where the stay S is supported on the post P. Then the force of compression of the stay will be $S = \dfrac{W\,l\,L}{m\,n}$.

The force of tension t on the gib will be $t = \dfrac{W\,l}{n}$.

The forces F, S and t must be expressed by the same units as that of the weight W.

The weight of the materials in the crane is not included in the formulas.

The tension of the chain is found by the formulas for pulleys.

The strain of the stays which hold the cap b is equal to the force F, or

$$F = \frac{W\,l}{h}.$$

The vertical pressure in the shoe a is equal to the sum of the weight W and weight of the crane.

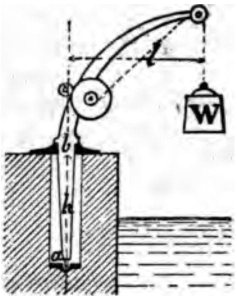
For foundry cranes the block c is moved in or out to suit the location of the weight to be filled, and the gib G and stay S are both made double, so that the chain can pass between the parts.

The lateral strain z on the gib where it bears on the stay will be

$$z = \frac{W\,l}{m}.$$

WHARF CRANES.

Fig. 45.



Wharf cranes for loading and unloading boats are often constructed like Fig. 45, and for which the static elements are the same as those for the foundry crane. When the direction of the chain on which the weight W hangs is parallel with post a, b, the strain F at b is equal to the horizontal pressure at a.

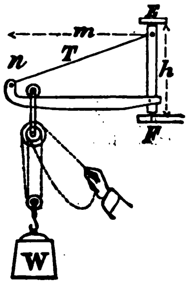
Static momentums $F\,h = W\,l$.

$$F = \frac{W\,l}{h}, \quad W = \frac{F\,h}{l}, \quad h = \frac{W\,l}{F}, \quad l = \frac{F\,h}{W}.$$

The lateral strength of the curved part of the post must compensate the stay in the foundry crane.

SHOP CRANE.

Fig. 46.



This form of cranes is used with differential pulleys. The tension-rod or tie T serves the same purpose as the stay in the foundry crane.

L = length of the tension-rod.

T = force of tension.

F = force or pressure in the post journals.

f = force of compression of the gib.

l = distance between the centres of the block and the post.

Static momentums $F\,h = W\,l$.

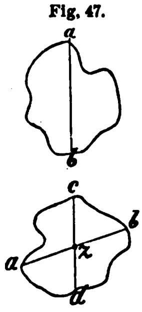
$$F = \frac{W\,l}{h}, \quad W = \frac{F\,h}{l}, \quad h = \frac{W\,l}{F},$$

$$l = \frac{F\,h}{W}, \quad T = \frac{W\,L\,l}{m\,n}, \quad f = \frac{W\,l}{n}.$$

CENTRE OF GRAVITY.

The centre of gravity of a body, or of a rigid system of bodies, is a point in which, if there suspended, the body will be in equilibrium in any position it may be placed, like that of a wheel or circleplane suspended in the centre.

Fig. 47.



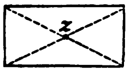
A body suspended freely from any point a will hang with its centre of gravity in the vertical line $a\ b$. Now suspend the body from another point c, and the centre of gravity will be on the line $c\ d$; then when the centre of gravity is on both the lines $a\ b$ and $c\ d$, it must evidently be at z, where the two lines cross one another.

The lines $a\ b$ and $c\ d$, or the centre of gravity z, can also be found by balancing the body on a sharp edge.

The centre of gravity of any figure or body is thus found by suspending or balancing the same in two different positions.

CENTRE OF GRAVITY OF A PARALLELOGRAM.

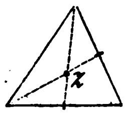
Fig. 48.



The centre of gravity of a square, rectangle or parallelogram is the point where the two diagonals cross one another.

TO FIND THE CENTRE OF GRAVITY OF A TRIANGLE.

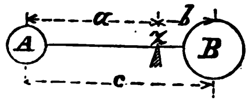
Fig. 49.



Bisect either two of the three sides of the triangle, and draw the dotted lines to the opposite angles, as shown by Fig. 49. The crossing of these lines is the centre of gravity z, which is one-third of the dotted line from the side.

CENTRE OF GRAVITY OF A SYSTEM OF TWO BODIES.

Fig. 50.



Let two bodies A and B be placed at a distance c between their centres of gravity, and z = centre of gravity of the system; then

$$A : B = b : a, \qquad \text{and } A\ a = B\ b.$$

$A\ a$ and $B\ b$ are moments of gravity, which are alike when the system is supported in its centre of gravity.

$$c = a + b, \qquad a = c - b \qquad \text{and } b = c - a.$$

$$A\,a = B\,b = B(c-a) = B\,c - B\,a.$$

$$\text{Then,} \quad A\,a + B\,a = B\,c, \quad \text{or} \quad a(A+B) = B\,c.$$

$$a = \frac{B\,c}{A+B}, \quad \text{and} \quad b = \frac{A\,c}{A+B}.$$

Example. $A = 6$ pounds, $B = 10$ pounds, and the distance $c = 18$ inches. Required the distance a?

$$a = \frac{B\,c}{A+B} = \frac{10 \times 18}{6+10} = 11.25 \text{ inches.}$$

CENTRE OF GRAVITY

of a system of any number of bodies placed in a straight line $o\,p$.

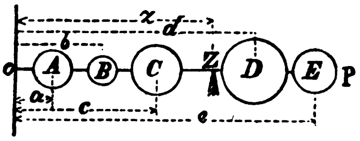
Assume any point o from which to refer the different moments of gravity, $Z = A + B + C + D + E$, or the sum of the weights of the different bodies; z = distance from o to the centre of gravity of the system.

Then the moments of gravity are

$$Z\,z = A\,a + B\,b + C\,c + D\,d + E\,e,$$

$$z = \frac{A\,a + B\,b + C\,c + D\,d + E\,e}{Z}.$$

Fig. 51.



A, B, C, etc. are weight, and a, b, c, etc. are the levers of the respective momentums of gravity.

CENTRE OF GRAVITY

of a system of bodies irregularly placed, with the centres of gravity in one plane.

Draw from o the lines $o\,p$ and $o\,q$, to form rectangular co-ordinate axes outside of the system. The right-angular moments of gravity are referred to the respective axes, as will be understood by the illustration.

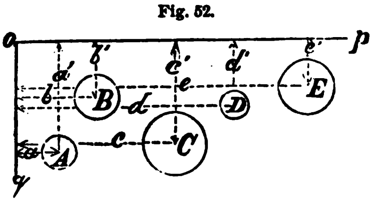
$$z = \frac{A\,a + B\,b + C\,c + D\,d + E\,e}{Z}.$$

From o set off the distance z toward p and draw the ordinate z'.

$$z' = \frac{A\,a' + B\,b' + C\,c' + D\,d' + E\,e'}{Z},$$

which gives the centre of gravity of the system.

Fig. 52.



By this method the centre of gravity of a great variety of systems of bodies can be ascertained; for instance, that of a vessel or steamboat. The axis $o\,p$ is drawn above the deck, and $o\,q$ at the stern of the vessel. A, B, C, etc. may represent weights of the propeller, engine, boiler, coal and cargo. The centre of gravity of each part is ascertained separately.

When a steamer is constructed, the weight and moments of gravity of each part ought to be calculated and summed up to correspond with the displacement, for which the following form of table is set up.

ELEMENTS OF THE STEAMER SHOOTING STAR.

Horizontal axis, 20 feet above load water-line.
Vertical axis, 10 feet aft the centre of rudder.

Details.	Weights of the parts.	Moments of gravity. Horizontal.		Vertical.	
		Lever.	Moment.	Lever.	Moment.
	Tons.	Feet.	Foot-tons.	Feet.	Foot-tons.
Rudder	2.1	9	18.9	26.5	55.6
Propeller shaft	12.8	47	601.6	27.5	352.6
Engines	50.1	75	3757.5	21	1052.1
Boilers with water	48	110	5280.0	23	1104.0
Coal	560	120	76200.0	25	14000.0
Cargo aft	500	50	25000.0	24	12000.0
Cargo forward	800	150	120000.0	30	24000.0
Propeller	3	27.5	82.5	13	239.0
	2102		248940.5		52803.3

Horizontal centre of gravity, $\frac{248940.5}{2102} = 115.1 - 10 = 105.1$ feet from centre of rudder.

Vertical centre of gravity, $\frac{52803.3}{2102} = 25.1 - 20 = 5.1$ feet below load water-line.

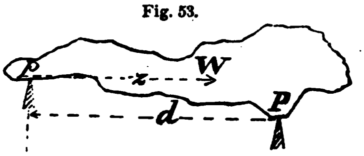
In practice the calculation is carried out with more details.

The horizontal position of the centre of gravity is required for knowing how the vessel will float in regard to the keel and load water-line.

The vertical position of the centre of gravity is required for determining the stability of the vessel.

TO FIND THE CENTRE OF GRAVITY OF ANY IRREGULAR BODY RESTING ON TWO SUPPORTS.

Fig. 53.



Having given the weights p and P bearing on the supports, the sum of which is equal to the weight W of the body, to find the horizontal distance z of the centre of gravity from the support p.

$$W : P : d : z \quad \text{and} \quad z = \frac{P\,d}{W}.$$

CENTRE OF GRAVITY BY THE CALCULUS.

By the aid of the calculus the centre of gravity of regular figures can be determined with precision.

Any figure or body can be considered as a system of bodies composed of its parts. The sum of the moments of gravity of all the parts is equal to the moment of gravity of the body or system.

Fig. 54.

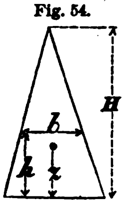


Fig. 54 represents a triangle with the base B and height H. Any element b of the triangle multiplied by the height h is the moment of gravity of that element.

Let A denote the area of the triangle, and z = height of centre of gravity from B. Then we have

$$z\,\partial A = b\;h\;\partial h.$$

$$b : B = (H-h) : H, \quad \text{and} \quad b = \frac{B(H-h)}{H}.$$

$$z\,\partial A = \frac{B\,(H-h)}{H}\,h\,\partial h. \qquad z\,A = \int \frac{B\,(H-h)}{H}\,h\,\partial h.$$

$$z\,A = \frac{B\,H^2}{2} - \frac{B\,H^2}{3} = B\,H^2\,(\tfrac{1}{2}-\tfrac{1}{3}).$$

$$z = \frac{B\,H^2}{A}\,(\tfrac{1}{2}-\tfrac{1}{3}).$$

$$A = \frac{B\,H}{2}, \qquad z = \frac{2\,B\,H^2}{B\,H}\,(\tfrac{1}{2}-\tfrac{1}{3}) = 2\,H\,(\tfrac{1}{2}-\tfrac{1}{3}).$$

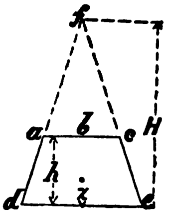
$$z = \tfrac{1}{3}\,H.$$

That is to say, the height of the centre of gravity from the base is equal to one-third the height of the triangle.

CENTRE OF GRAVITY OF A QUADRANGLE.

The illustration represents a quadrangle, a, c, d, e, of which the top b is parallel with the base B. Prolong the sides $d\ a$ and $e\ c$ until they meet at f, which will form the triangle d, f, e of height H.

Fig. 55.



h = height of the quadrangle.

A = area of the quadrangle.

O = area of the triangle a, f, c.

Q = area of the whole triangle d, f, e.

Height of centre gravity of the whole triangle is $\frac{1}{3}H$.

Height of centre gravity of triangle a, f, c is $\frac{1}{3}(H-h)+h$.

z = height of centre gravity of the quadrangle.

Moment of gravity, $Q(\frac{1}{3}H) = A\,z + O[\frac{1}{3}(H-h)+h]$. . . . 1.

$A\,z = \frac{1}{3}Q\,H - O(\frac{1}{3}H+\frac{2}{3}h)$. . . . . 2.

Height H, $\quad H:(H-h) = B:b, \quad H = \dfrac{B\,h}{B-b}$. . . . 3.

Area of quadrangle, $A = \dfrac{h}{2}(B+b)$. . . . . . . 4.

Area of triangle, a, f, c, $\quad O = \dfrac{b}{2}(H-h) = \dfrac{b}{2}\left(\dfrac{B\,h}{B-b} - h\right)$. . . . 5.

Area of triangle, d, f, e, $\quad Q = \dfrac{B\,H}{2} = \dfrac{B^2h}{2(B-b)}$. . . . . 6.

Moment of gravity, $A\,z = \frac{1}{3}Q\,H - \frac{1}{3}O\,H - \frac{2}{3}O\,h$. . . . 7.

Insert the values 3, 4, 5 and 6 for the corresponding quantities H, A, O and Q in Formula 7.

$$A\,z = \tfrac{1}{3}.\frac{B^2h}{2(B-b)}.\frac{B\,h}{(B-b)} - \tfrac{1}{3}\frac{b}{2}\left(\frac{B\,h}{B-b} - h\right)\frac{B\,h}{B-b} - \frac{2\,b\,h}{3\times 2}\left(\frac{B\,h}{B-b} - h\right).$$

This formula reduces itself to $z = \dfrac{h(B+2\,b)}{3(B+b)}$, which gives the height of centre of gravity of the quadrangle above the base B.

CENTRE OF GRAVITY OF A CONE.

D = diameter of the base of the cone.

h = height, x and y are co-ordinates.

$$x : y = h : D, \qquad y = \frac{D\,x}{h}.$$

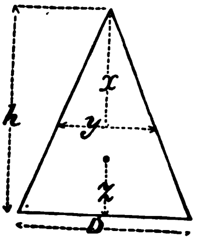
Volume of cone, $C = \dfrac{\pi\,D^2 h}{12}$.

$$d\,\partial C = \frac{\pi\,y^2}{4}\,x\,\partial x = \frac{\pi\,D^2 x^3}{4\,h^2}\,\partial x.$$

$$d\,C = \frac{\pi\,D^2}{4\,h^2}\int x^3 \partial x = \frac{\pi\,D^2 h^4}{4\,h^2 4} = \frac{\pi\,D^2 h^2}{16}.$$

$$d.\frac{\pi\,D^2 h}{12} = \frac{\pi\,D^2 h^2}{16}, \qquad d = \tfrac{3}{4}h, \qquad z = \tfrac{1}{4}h.$$

Fig. 56.



CENTRE OF GRAVITY OF A PARABOLA.

The illustration represents a parabolic plane of abscissa x and ordinate y.

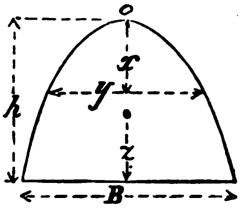
z = height of centre of gravity above the base B.

d = depth of centre of gravity from the vertex o, or $h = d + z$ and $z = h - d$.

The formula for a parabola is $y = 2\sqrt{p\,x}$.

The parameter of a parabola is $p = \dfrac{B^2}{4\,h}$.

Fig. 57.



The area of a parabola is $A = \tfrac{2}{3}B\,h$.

The moment of gravity from the vertex is

$$d\,\partial A = y\,x\,\partial x = 2\sqrt{p\,x}\;x\,\partial x = 2\sqrt{\frac{B^2 x}{4\,h}}\;x\,\partial x.$$

$$d\,\partial A = \frac{B}{\sqrt{h}}\,x^{1.5}\partial x, \qquad d\,A = \frac{B}{\sqrt{h}}\int x^{1.5}\partial x = \frac{B\,h^{2.5}}{\sqrt{h}\,2.5}.$$

$$d\tfrac{2}{3}B\,h = \frac{B\,h^2}{2.5}, \qquad d = \tfrac{3}{5}h, \text{ and } z = \tfrac{2}{5}h.$$

That is to say, the height of the centre of gravity of a parabolic plane is two-fifths of the height of the parabola.

CENTRE OF GRAVITY OF A PARABOLOID.

Meaning of letters is the same as in the preceding example.

Volume of a paraboloid $C = \dfrac{\pi B^2 h}{8}$.

Moment of gravity for solidity of the paraboloid is

$$d\,\partial C = \frac{\pi}{4} y^2 x\,\partial x = \pi p x^2\,\partial x.$$

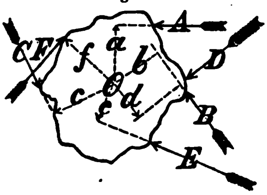
$$d\,\partial C = \frac{\pi B^2}{4h} x^2\,\partial x, \quad d\,C = \frac{\pi B^2}{4h}\int x^2\,\partial x = \frac{\pi B^2 h^3}{12h}.$$

$$d\frac{\pi B^2 h}{8} = \frac{\pi B^2 h^2}{12}, \quad d = \tfrac{2}{3}h, \text{ and } z = \tfrac{1}{3}h.$$

The centre of gravity of a paraboloid is one-third of the height from the base.

EQUILIBRIUM OF STATIC MOMENTUMS.

Fig. 58.



The illustration represents a body fixed on an axis O, and several forces A, B, C and D, E, F are acting on their respective levers a, b, c and d, e, f to rotate the body in opposite directions.

Then the opposing static momentums will be in equilibrium, when

$$A\,a + B\,b + C\,c = D\,d + E\,e + F\,f.$$

When the sum of any number of static momentums acting to rotate a body in one direction, is equal to the sum of the momentums acting in opposite directions, then the opposing forces are in equilibrium, and the body will remain at rest.

When the sums of the opposing static momentums are not alike, the body will move with a momentum equal to the difference between the two sums.

DYNAMICS.

§ 1. **Dynamics** is that branch of mechanics which treats of *forces in* motion, producing *power* and *work*. It comprehends the action of all kinds of machinery, manual and animal labor in the transformation of physical work.

Quantity is any principle or magnitude which can be increased or diminished by augmentation or abatement of homogeneous parts, and which can be expressed by a number.

Element is an essential principle which cannot be resolved into two or more different principles.

Function is any compound result or product of two or more different elements.

A function is resolved by dividing it with one or more of its elements.

Force, Velocity and **Time** are simple physical elements.

Power, Space and **Work** are functions of those elements.

The combinations of the elements in the functions are as follows:

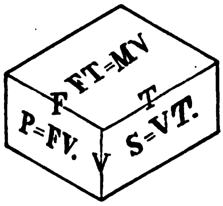
ELEMENTS.	FUNCTIONS.
Force $= F.$	Power $P = F\ V.$
Velocity $= V.$	Space $S = V\ T.$
Time $= T.$	Work $K = F\ V\ T.$
Mass $= M.$	Work $K = \frac{1}{2}\ M\ V^2.$

Fig. 59.

$F : M = V : T.$

MOMENTUM.

$M\ V = F\ T.$



$F : M = \frac{1}{2} V^2 : S.$

WORK.

$F S = \frac{1}{2} M\ V^2.$

These are the fundamental principles in dynamics.

DYNAMICS COMPARED WITH GEOMETRY.

§ 2. In geometry we have three fundamental elements, expressed by the terms **Length, Breadth** and **Thickness,** which serve to represent to the mind the nature of the several properties of geometrical space.

We have in like manner in dynamics three fundamental elements, expressed by the terms **Force, Velocity** and **Time,** which represent the nature of physical **Power, Space** and **Work.**

Force is any action which can be expressed simply by weight without regard to motion or time; it is an essential principle which cannot be resolved into two or more principles, and is therefore a simple element. *Force* is the first element in dynamics, and corresponds to length in geometry.

Velocity is speed or rate of motion, an essential principle which cannot be resolved into two or more principles, and is therefore a simple element. *Velocity* is the second element in dynamics, and corresponds to breadth in geometry.

Time is duration, or that measured by a clock; it is an essential principle which cannot be resolved into two or more principles, and is therefore a simple element. *Time* is the third element in dynamics, and corresponds to thickness in geometry.

Power is the product of the first and second elements, *force* and *velocity*, and is therefore a function.

Power is the first function in dynamics, and corresponds to the product of length and breadth, which is surface in geometry.

Space is the product of the second and third elements, *velocity* and *time*, and is therefore a function.

Space is the second function in dynamics, and corresponds to the product of breadth and thickness, which is the area of a cross-section of a solid in geometry.

Work is the product of the three simple elements *force*, *velocity* and *time*, and is therefore a function.

Work is the third function in dynamics, and corresponds to the product of length, breadth and thickness,which is volume in geometry.

Work is also the product of the element *force* and function *space*, because the function space contains the elements *velocity* and *time:* like volume in geometry, it is the product of length and area of cross-section.

Work is also the product of the function *power* and element *time*, because the function power contains the elements *force* and *velocity:* like volume in geometry, it is the product of area and thickness.

DETAILED EXPLANATIONS OF THE ELEMENTS AND FUNCTIONS.

FORCE.

§ 3. **Force** is any action which can be expressed simply by weight, and is distinguished by a great variety of terms, such as *attraction, repulsion, gravity, pressure, tension, compression, cohesion, adhesion, resistance, inertia, strain, stress, strength, thrust, burden, load, squeeze, pull, push, pinch, punch, etc.*, all of which can be measured or expressed by weight without regard to motion, time, power, or work.

All bodies in nature possess the incessant virtues of attracting and repelling one another, which action is recognized as force.

The physical constitution of force is not yet known, but it appears that the force of repulsion is derived from heat, and that of attraction from cold or absence of heat.

It is well known that bodies generally expand when heated, which proves the repulsive force of heat; and that bodies contract by decrease of temperature shows the superior action of the attractive force in the absence of heat.

The two forces act opposite to one another, and when the force of attraction is superior to that of repulsion, the body will maintain the form of solid. When the opposing forces are equal, the body will maintain the form of liquid, and when the force of repulsion is superior to that of attraction, the body will have the form of a gas.

The force of universal attraction will be explained in its proper place.

The unit for measuring force is any assumed weight, as pound or ton.

MOTION.

§ 4. **Motion** is a continuous change of position in regard to assumed fixed objects. Motion or rest are only relative; that is to say, when two bodies change their relative positions, either one of them can be considered at rest and the other in motion. There is no absolute rest known in the universe. Our earth revolves around its axis and also in its orbit around the sun, but the sun also revolves both around his axis and in a small irregular orbit in regard to his planetary system, whilst the whole planetary system moves bodily in space.

We are therefore not able to establish any absolute rest, but must consider motion and rest only as relative.

Motion is expressed by the following terms: *Move, going, walking, passing, transit, involution* and *evolution, run, locomotion, flux, rolling, flow, sweep, wander, shift, flight, current*, etc.

Motion of Translation is when a body moves with a uniform velocity in a straight line without revolving, or when each particle of the body moves in parallel lines.

Motion of Gyration is the same as rotation, or when all the particles in a body describe concentric circles around one common axis.

Helicoidal Motion is when a body revolves around an axis, and at the same time moves in the direction of that axis, like a screw or rifle-ball, which is the result of the two motions of translation and gyration.

Lateral Motion is that motion of translation which a body generates in a direction out of its greatest extension. Lateral motion may also be referred to a stationary line or plane from or to which a body is moving. Small rivers run into larger ones from lateral directions.

Rolling Motion is the combination of rotary motion of a body and lateral motion of the axis of rotation, like a wagon-wheel rolling on the ground.

There are also different kinds of motions, designated by the name of the path, of the curve, described by the motion, such as parabolic motion, elliptic motion, cycloidal motion, etc.

Space Motion. We say that a body has more or less motion in regard to greater or less space moved through, in which case motion means space.

VELOCITY.

§ 5. **Velocity** is rate of motion. It is obtained by dissolving the function *space* and eliminating the element *time*.

$$\text{Velocity} = \frac{S}{T} = \frac{V\,T}{T} = V, \text{ the element.}$$

Velocity is independent of *space* and *time*, but in order to obtain its value or expression as a quantity, we compare space with time. Thus, when the value of velocity of a moving body is required, we measure a space which the body passes through, and divide that space with the time of passage, and the quotient is the velocity.

Velocity is therefore expressed by space per time, as feet per second or miles per hour. A definite velocity can be expressed by any units of space and time, because velocity is an essential principle which cannot be resolved into two or more principles. A velocity of 22 feet per second is equal to a velocity of 15 miles per hour. For a definite velocity we can reduce the space and time to infinitely small, or say

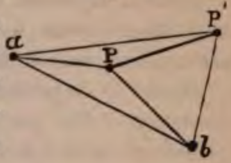
absolutely nothing, without affecting the velocity in the least; which proves that velocity is an independent virtue or an element.

Absolute Velocity is that measured or observed in the real path of motion.

Apparent Velocity is that observed from an assumed fixed point, and which can be measured only as angular velocity.

The terms velocity and motion are often used for one another, and neither one of them can exist without the other, because it is velocity which generates motion by the aid of time.

Fig. 60.



Suppose a body to move from P to P', an observer at a would see less velocity or motion than would one at b; but both motions are apparent velocities, which can be measured only by the angles $P\,a\,P'$ and $P\,b\,P'$, whilst the absolute velocity must be measured in the path $P\,P'$.

Velocity or rate of motion is expressed by a variety of terms, as follows:

QUICK MOTION.

Speed, swiftness, rapidity, fleetness, speediness, quickness, haste, hurry, race, forced march, gallop, trot, run, rush, scud, dash, spring, etc.

SLOW MOTION.

Slowness, tardiness, dilatoriness, slackness, drawl, retardation, hobbling, creeping, lounging, linger, sluggish, crawl, drawl, loiter, glide, languid, drowsy, etc.

Velocity is variously expressed by different units of length and time; for example, in machinery it is generally estimated in feet per second or minute; in steamboat and railway traveling, by miles per hour; the velocity of light and electricity, by miles per second; whilst the retirement of the Niagara Falls toward Buffalo, and the sinking and rising of land or sea may be expressed in feet or inches per century.

The velocity of light in planetary space is about 200,000 miles per second, or the same as that of electricity through good conductors. The velocity of the earth in its orbit around the sun in reference to a fixed star is about 19 miles per second.

The velocity of a point on the earth's equator in reference to the sun is about 1037 miles per hour, or 1520 feet per second.

Angular velocity is an apparent motion referred to a fixed centre, like that of a revolving wheel or an oscillating pendulum; it is measured by periodical revolutions or oscillations, generally denoted by the letter n.

6

TIME.

§ 6. **Time** implies a continuous perception, recognized as duration.

Chronology is the science of time.

Instant and moment are points of time.

Epoch is the beginning of any time marked with some remarkable events and recorded by historians or chronologists. Era is nearly the same as epoch, except that it is generally fixed by nations or denominations, as the Christian era.

Time is expressed by a great variety of units—namely, *millennium*, a thousand years; *century*, one hundred years; *score*, twenty years; *year*, *season*, *month*, *fortnight*, *week*, *day*, *hour*, *minute* and *second*.

Cycle is a period of time in which similar phenomena of the heavenly bodies perpetually occur, such as the cycle of the sun, a period of twenty-eight years, at which the days of the week return to the same dates of the month.

The most ordinary unit for measuring time is derived from the period of one revolution of the earth around its axis, in reference to the sun, which is called a solar day, and divided into 24 hours, each hour 60 minutes and each minute 60 seconds.

Another unit of time is the period occupied by the earth in making one revolution around the sun, in reference to an assumed fixed star, which unit is called a sidereal year, and contains 365 days, 6 hours, 9 minutes and 9.6 seconds, mean solar time.

We have no positively fixed standard for measuring time, for the period of one revolution of the earth around its axis, as well as that around the sun, are both liable to changes by meteors or heavenly bodies falling on the earth which accelerate or retard its motion. Such changes have, within the time of our astronomical records, been so small as to amount to probably not more than a few seconds in thousands of years; but it may happen at any time that a heavenly body large enough might strike the earth and cause a change of time which would at once be perceived by us all without the aid of instruments for that purpose. (See *Astronomy*.)

POWER.

§ 7. **Power** is the product of force and velocity; that is to say, a force multiplied by the velocity with which it is acting, is the power in operation.

The English unit for measuring power is *a force of one pound acting with a velocity of one foot per second*, and called *one foot-pound of power*, or *one effect*.

Man-power is a unit of power established by MORIN to be equivalent to 50 foot-pounds of power, or 50 effects; that is to say, a man turning a crank with a force of 50 pounds and with a velocity of one foot per second is a standard man-power.

An ordinary workingman can exert this power eight hours per day without overstraining himself.

Horse-Power is a unit of power established by James Watt, to be equivalent to a force of 33,000 pounds acting with a velocity of one foot per minute, which is the same as a force of 550 pounds acting with a velocity of one foot per second.

That is to say, one horse-power is 550 foot-pounds of power or effects, or 11 man-power of 50 effects each.

The product of any force in pounds and its velocity in feet per second, divided by 550, gives the horse-power in operation.

In Watt's rule for horse-power is given a velocity of only one foot per minute, which is equal to 0.2 or $\frac{1}{5}$th of an inch per second—about the velocity of a snail. The force corresponding to this velocity is 33,000 pounds, or about 15 tons, which is too large for a clear conception of its magnitude, and a horse can never pull with such a force. A horse can pull 550 pounds with a velocity of one foot per second, which is the most natural expression for horse-power. This expression is used on the continent of Europe.

Foreign Terms and Units for Horse-Power.

Countries.	Terms.	Eng. translation.	Units.	Eng. equivalent.
English..........	Horse-power.	Horse-power.	550 foot-pounds.	550 foot-lbs.
French...........	Force de cheval.	Force-horse.	75 kilogr. metres.	542.47 foot-lbs.
German.........	Pferde-krafte.	Horse-force.	513 Fuss-funde.	582.25 foot-lbs.
Swedish.........	Häst-kraft.	Horse-force.	600 skalpund-fot.	542.06 foot-lbs.
Russian.........	Syl-lochad.	Force-horse.	550 Fyt-funt.	550 foot-lbs.

The quantities **Force** and **Power** are clearly distinguished by different terms only in the English language.

On the continent of Europe horse-power is called **horse-force** or **force-horse**, which does not distinguish force from power.

Horse-force can be considered to be the force with which a horse can pull.

The word *force* is needed in the Swedish and German languages.

Puissance, in the French language, means *power*, but the term is not generally used in that sense in dynamics.

In most of the continental languages there are words which cor-

respond with power, but they are not used in that sense in dynamics, where the term force is used for power.

The Swedish and German word *kraft* ought to be used for power only, and not for *force*, as it is also used.

The words expressing *work* are clear and definite in all languages.

EFFECT.

The term *effect* has been used to denote both foot-pounds of power and foot-pounds of work. These two kinds of foot-pounds have heretofore not been clearly distinguished from one another, for which reason the term effect will hereafter be used only to denote foot-pounds of power.

P = simple power in foot-pounds or effects.

F = force in pounds.

V = Velocity in feet per second.

HP = Horse-power.

Simple power $P = F\,V$. Effects.

$$\text{Horse-power HP} = \frac{P}{550}.$$

$$\text{Horse-power HP} = \frac{F\,V}{550}.$$

$$\text{Man-power} = \frac{P}{50}.$$

$$\text{Man-power} = \frac{F\,V}{50}.$$

One horse-power = 11 man-power.

Any action of force producing motion is power, which is independent of time.

In lifting a weight vertically the force F is equal to the weight lifted, but in drawing a load on a road the force of traction may be considerably less than the weight of the load.

A weight of 1000 pounds lifted vertically with a velocity $V = 2$ feet per second requires a power of 2000 effects, which is equal to $2000 : 50 = 40$ man-power, or $2000 : 550 = 3.63$ horse-power.

A load of 1000 pounds drawn on a horizontal road with a velocity $V = 2$ feet per second may require a tractive force of only $F = 100$ pounds, and the power will be 200 effects, or 4 man-power, which is only one-tenth of the power required in lifting the same weight vertically.

Power is the differential of work, or any action which produces work, whether mental or physical.

Power multiplied by the time of action is work—work divided by time is power. Writers on dynamics have heretofore assumed that "*power is the work done in a unit of time,*" which is an error.

The number which expresses the work done in a unit of time, is equal to the number which expresses the power in operation; but that does not prove the two quantities to be alike.

When we say "in a certain time," which is equivalent to the expression "per unit of time," we divide by the time.

Work is the product of the three elements **Force, Velocity** and **Time,** and when we say "work per unit of time," we eliminate the time from the work, and the remainder is *power*, which is the product of *force* and *velocity*.

Power may be expressed by the following terms:

Traction, *propulsion*, *impulsion*, *capability*, *puissance*, *labor*, *haul*, *drag*, *draw*, *heave*, *occupation*, *activity*, *vigor*, *energy*, *etc.*, or any action which implies force and motion without regard to time.

SPACE.

§ 8. **Space** in dynamics means linear space, which is a function of the second and third elements, velocity V and time T, and may be likened to the cross-section of a solid, which is a function of breadth and thickness.

Space is herein denoted by

$$S = V\,T,$$

which means that the space S, expressed in linear feet, is the product obtained by multiplying together the velocity V and time T.

Space cannot be generated or conceived without the two elements motion and time.

In viewing a short linear space our mind flies so rapidly over it that we miss the conception of motion and time. The length of a piece of wood has been generated by the motion and time required for its growth, and when our mind surveys that length, motion and time are required for passing from one end of it to the other.

In like manner, when we at a glance survey a bridge we are unable to appreciate its length without following it in contemplation from pier to pier, with some expenditure of time and velocity from one end to the other.

So also when we imagine objects or cities far distant apart, our conceptions of the magnitude of the distances between them are exces-

sively vague and inexact without an imaginary transit over them from point to point, which also requires both time and velocity.

The distance run by a locomotive or steamboat is the product of the velocity and time of the trip, and no distance can be accomplished without either of these elements.

Geometrical spaces are magnitudes of three different kinds—namely, *linear*, *superficial* and *voluminous*.

Linear space is that generated by the product of time and motion of a point.

Superficial space is that generated by the product of velocity and time of lateral motion of a line.

Voluminous space is that generated by the product of velocity and time of lateral motion of a plane.

In determining velocity it appears as if motion were dependent on space and time, because we measure a space and divide it by the time, in order to form a conception of velocity or the rate of motion. Space is the product of time and velocity, and when we divide that product by time, the quotient will be the simple element velocity or rate of motion.

In other words, when we divide the space with time we resolve the function space into its constituent elements and eliminate the time, and the quotient is the simple element velocity. If we divide the space with velocity, the latter is eliminated from the former, and the quotient is the simple element time.

Space in dynamics means the *generation* of that space by velocity and time. A line of any kind cannot be drawn without velocity and time.

A locomotive running with a uniform velocity of 30 miles per hour will make 2640 feet per minute or 44 feet per second; and if we diminish the space and time to infinitely small, or, say, absolutely nothing, the velocity of 30 miles per hour is still constant when passing that time and space reduced to a point.

In geometry length is an element without regard to velocity or time, but in dynamics linear space means a physical function generated by velocity and time.

Length is a geometrical element.

Space is a physical function.

In regard to space being composed of velocity and time, the following question has been asked: The distance to the moon is space; what has that to do with velocity and time? The moon has never been on the earth, and consequently not moved from it with velocity and time? The answer is as follows: In order to find the distance to the

moon, a space is measured on the earth's surface, and is obtained by velocity and time, which is converted into a base-line—namely, the diameter of the earth. The moon's horizontal parallax is next measured, through which the velocity and time in the base-line is multiplied until it reaches the moon. Thus, the space to the moon is composed of velocity and time.

WORK.

§ 9. **Work** is the product obtained by multiplying together the three elements, *force* F, *velocity* V and *time* T, or *work* $K = F\ V\ T$.

Work may also be expressed by $K = F\ S$, or the product obtained by multiplying together the *force* F and *space* S, in which it appears as if work was independent of *time;* but the *time* is included in the space $S = V\ T$. A given amount of work may be performed in any desired length of time, but the work is nevertheless dependent on whatever time consumed in its execution.

A definite quantity of work is not confined to any definite ratio or relation to either of its constituent elements, for either one or two of them may vary *ad libitum*, but only at the expense of the remaining two or one. A definite quantity at work, only requires a definite product of the combined actions of the three elements. *Work* is thus dependent on time as well as on force and velocity, for without either one of these three elements it ceases to be work.

If work was independent of time, then any amount of work could be accomplished in no time.

The greatest amount of work known to have been accomplished in the shortest time, is that in the explosion of nitro-glycerine, which is instantaneous to our perception; but it required time notwithstanding.

Work may also be expressed by $K = P\ T$, or the product of power and time.

The work of a steam-engine operating with a constant power, will be directly as the time of operation, and so with all labor, whether it be mechanical or manual.

The longer we toil the more work will be done, but if we have no time to do the work it will remain undone.

Much of the confusion in dynamics has arisen from misconception of the difference between specific quantities and abstract numbers.

When a quantity is multiplied by an abstract number, the product will be the same as the sum of so many concrete quantities added together as indicated by the number, and the operation will change the magnitude, but not the nature, of that quantity. But when a

quantity is multiplied by another quantity, the product will be a third quantity of different nature from that of its constituent quantities.

A force of 2 pounds working with a velocity of 3 feet per second is a power of 6 foot-pounds, which, multiplied by the abstract number 4, will be a power of 24 foot-pounds. But the same power, 6 foot-pounds, multiplied by the quantity 4 seconds, will be 24 foot-pounds of work.

Three square feet multiplied by the abstract number 2 will be 6 square feet, but when three square feet are multiplied by a thickness of 2 feet the product will be 6 cubic feet.

The erroneous expression that "power is the work done in a unit of time" implies that power is a portion of work. Power multiplied by a unit of time is work, and work divided by a unit of time is power. In both these cases the unit of time does not change the numerical value of the quantities, but it changes their nature from one to the other.

A pool of water, say 1000 square feet of surface, is frozen over with ice one foot thick, and there will consequently be 1000 cubic feet of ice in the pool, which though identical in number with that of the surface, does not prove that a square foot is a cubic foot.

The surface of the pool represents *power;* the thickness of the ice represents *time*, and the volume of the ice represents the *work* consumed in freezing it.

UNITS OF WORK.—FOOT-POUND.

The English unit of work is assumed to be that accomplished by a force of one pound raising an equal weight one foot high, which unit is called a **foot-pound.** Then a force of 6 pounds working through a space of 4 feet is equivalent to 24 foot-pounds of work.

This unit is very convenient for small amounts of work, but it is too small for many purposes in practice.

FOOT-TON.

English ordnance officers have adopted a larger unit for work, namely, **foot-ton,** which is used for expressing work of heavy ordnance. It means the work of lifting one ton one foot high.

WORKMANDAY.

A laborer working eight hours per day can exert a power of 50 foot-pounds. A day's work will then be $50 \times 8 \times 60 \times 60 = 1{,}440{,}000$ foot-pounds of work, which may be termed a *workmanday.*

All kinds of heavy work can be estimated in workmandays, such as the building of a house, a bridge, a steamboat, canal and railroad excavations and embankments, loading or unloading a ship, powder and steam-boiler explosions, and the capability of heavy ordnance, etc.

The magnitude of the unit *workmanday* is easily conceived, because it is that amount of work which a laborer can accomplish in one day. Work expressed in foot-pounds, divided by 1,440,000, gives the work in *workmandays*.

A work of 20 workmandays can be accomplished by 20 men in one day, by one man in 20 days, by 4 men in 5 days, or by 10 men in two days.

§ 10. DIFFERENT KINDS OF FOOT-POUNDS.

There are four different kinds of foot-pounds in mechanics—namely,

1st. Foot-pounds of static momentum, which are force in pounds multiplied by its lever of action in feet.

2d. Foot-pounds of dynamic momentum are mass expressed in pounds, multiplied by velocity in feet per second.

3d. Foot-pounds of power (effects) are force in pounds, multiplied by velocity in feet per second.

4th. Foot-pounds of work are force in pounds, multiplied by space in feet.

It will be observed that foot-pounds of static momentum and foot-pounds of work are both the product of force and linear space, from which it would appear that these two functions are substantially alike, but they are of entirely different nature.

Static momentum is force multiplied by the geometrical element length, without regard to velocity and time; in which case the force has nothing to do with the generation of that length.

Work is force multiplied by the physical function space, which is generated by the two elements velocity and time.

Work done is expressed by the following terms:

Hauled, *dragged*, *raised*, *heaved*, *cultivated*, *tilted*, *broken*, *crushed*, *thrown*, *wrought*, *fermented*, *labored*, *embroidered*, etc., or any expression which implies the three simple elements, *force*, *velocity* and *time*.

Power is the differential of work.

Work is the integral of power.

The following formulas show the different combinations of the dynamic elements and functions. Should either or both the force and the velocity be variable or irregular, the mean action in the time T must be inserted, and the formulas will answer for any kind of operation.

§ II. DYNAMICAL FORMULAS.

Force or Pressure in Pounds.

$F=\frac{P}{V}$.	1	$F=\frac{K}{S}$.	3
$F=\frac{550\ \text{HP}}{V}$.	2	$F=\frac{K}{V\,T}$.	4

Velocity in Feet per Second.

$V=\frac{S}{T}$.	5	$V=\frac{550\ \text{HP}}{F}$.	7
$V=\frac{P}{F}$.	6	$V=\frac{K}{F\,T}$.	8

Time of Action in Seconds.

$T=\frac{S}{V}$.	9	$T=\frac{F\,S}{550\ \text{HP}}$.	11
$T=\frac{F\,S}{P}$.	10	$T=\frac{K}{F\,V}$.	12

Power in Effects.

$P=F\,V$.	13	$P=550\ \text{HP}$.	15
$P=\frac{F\,S}{T}$.	14	$P=\frac{K}{T}$.	16

Space Passed Through in the Time T.

$S=V\,T$.	17	$S=\frac{550\ T\,\text{HP}}{F}$.	19
$S=\frac{P\,T}{F}$.	18	$S=\frac{K}{F}$.	20

Horse-Power.

$\text{HP}=\frac{P}{550}$.	21	$\text{HP}=\frac{F\,S}{550\ T}$.	23
$\text{HP}=\frac{F\,V}{550}$.	22	$\text{HP}=\frac{K}{550\ T}$.	24

Work in Foot-Pounds.

$K=F\,V\,T$.	25	$K=F\,S$.	27
$K=P\,T$.	26	$K=550\ \text{HP}\ T$.	28

It will be observed in the preceding formulas that an element is never divided by an element, but a function is divided by an element only when that function contains the element divided with.

Power divided by velocity gives force, because power contains the elements force and velocity; but power cannot be divided by time, because time is not a constituent element of power.

Work can be divided by either one or two of its three constituent elements. When work is divided by either two of its elements, the product will be the third element.

Different elements or functions cannot be added to or subtracted from one another. Power or space cannot be added to or subtracted from work. Force, velocity or time cannot be added to or subtracted from space.

When a formula contains several terms, all the terms must be of the same kind; for instance:

$$\text{Work } K = T\left(F\,V + P - \frac{K}{T}\right).$$

The terms within the parentheses are all power, which multiplied by time gives work.

Mistakes in dynamical formulas are easily detected by the above rules.

No element can be converted into an element of a different kind.

§ 12. EXAMPLES CORRESPONDING WITH THE FORMULAS.

Force or Pressure in Pounds.

Example 1. A power $P = 6400$ effects is operating with a velocity of $V = 12$ feet per second. Required the force F?

$$F = \frac{P}{V} = \frac{6400}{12} = 533 \text{ pounds.}$$

Example 2. The piston of a steam-engine of $\text{HP} = 24$ horses is moving at the rate of $V = 8$ feet per second. Required the force F?

$$F = \frac{550\ \text{HP}}{V} = \frac{550 \times 24}{8} = 1650 \text{ pounds.}$$

Example 3. A work of $K = 3266$ foot-pounds is accomplished in a space $S = 16$ feet. Required the force F?

$$F = \frac{K}{S} = \frac{3266}{16} = 204 \text{ pounds.}$$

Example 4. A work of $K=183600$ foot-pounds was accomplished with a velocity $V=18$ feet per second in a time of 3 minutes, or $T=3\times 60=180$ seconds. Required the force F?

$$F=\frac{K}{VT}=\frac{183600}{18\times 180}=56.6 \text{ pounds.}$$

Velocity in Feet per Second.

Example 5. A body moves through a space of $S=160$ feet in a time of $T=40$ seconds. Required the velocity V?

$$V=\frac{S}{T}=\frac{160}{40}=4 \text{ feet per second.}$$

Example 6. A power of $P=4266$ effects is operating with a force $F=760$ pounds. Required the velocity V?

$$V=\frac{P}{F}=\frac{4266}{760}=5.6 \text{ feet per second.}$$

Example 7. The cylinder of a steam-engine of $H\!P=160$ horse-power is 24 inches in diameter, and the effective steam-pressure is 30 pounds to the square inch. Required the velocity of the steam-piston?

The area of the piston is 452.39 square inches, which multiplied by 30 pounds to the square inch will be a force of

$$F=13570.8 \text{ pounds.}$$

$$V=\frac{550\ H\!P}{F}=\frac{550\times 160}{13570.8}=6.5 \text{ feet per second.}$$

Example 8. A work of $K=864360$ foot-pounds is accomplished with a force of $F=68$ pounds in a time of 5 minutes. Required the velocity V?

The time $T=5\times 60=300$ seconds.

$$V=\frac{K}{FT}=\frac{864360}{68\times 300}=42.4 \text{ feet per second.}$$

Time of Action in Seconds.

Example 9. A space of $S=2896$ feet is generated with a velocity of $V=25$ feet per second. Required the time T?

$$T=\frac{S}{V}=\frac{2896}{25}=115.84 \text{ seconds.}$$

Example 10. A force of $F=4596$ pounds is working through a space $S=960$ feet. What time is required for the force to generate a power of $P=840680$ effects?

$$T=\frac{FS}{P}=\frac{4596\times 960}{840680}=5.25 \text{ seconds.}$$

Example 11 a. The stroke of a steam-piston is four feet, and the effective pressure of steam is $F=46360$ pounds. The power of the engine is $\text{HP}=500$ horses. What time is required of the engine to make 64 double strokes?

The space $S=4\times 2\times 64=512$ feet.

$$T=\frac{FS}{550\ \text{HP}}=\frac{46360\times 512}{550\times 500}=86 \text{ seconds.}$$

Example 11 b. What time is required to raise a weight of 200 tons to a height of $S=50$ feet with an engine of $\text{HP}=8$ horse-power?

$$F=200\times 2240=448000 \text{ pounds.}$$

$$T=\frac{FS}{550\ \text{HP}}=\frac{448000\times 50}{550\times 8}=509 \text{ seconds,}$$

or 8 minutes and 29 seconds.

Example 12. What time is required to accomplish a work of $K=96236000$ foot-pounds, with a force $F=88$ pounds, moving with a velocity of $V=1.5$ feet per second?

$$T=\frac{96236000}{88\times 1.5}=729066 \text{ seconds,}$$

or 202 hours 31 minutes and 6 seconds.

Assuming a workmanday to be 1,440,000 foot-pounds, it would require about 67 such units to accomplish the work; that is to say, one man could do the work in 67 days, or 67 men could accomplish it in one day.

Power in Effects or Foot-Pounds.

Example 13. A weight of five tons is raised vertically at the rate of $1\frac{1}{2}$ inches per second. Required the power P?

The force $F=5\times 2240=11200$ pounds.

Velocity $V=0.125$ feet per second.

$$P=11200\times 0.125=1400 \text{ effects.}$$

One man-power is 50 effects, and it would require $1400:50=28$ men to raise five tons with a velocity of $1\frac{1}{4}$ inches per second at continued work.

One horse-power is 550 effects, and it would require $1400:550=2.55$ horse-power for the same work.

Example 14. What power is required to lift a weight of three tons a space of $S=5$ feet in a time of 10 minutes?

$$P=\frac{FS}{T}=\frac{3\times2240\times5}{10\times60}=56 \text{ effects.}$$

Example 15. How many effects are there in $\text{HP}=30$ horse-power?

$$P=550\times30=16500 \text{ effects.}$$

Example 16. What power is required to do a work of $K=186000$ foot-pounds in one minute? $T=60$.

$$P=\frac{186000}{60}=31000 \text{ effects.}$$

Space Passed Through in the Time T.

Example 17. A body moving with a velocity of $V=960$ feet per second for a time of $T=5$ seconds. Required the space passed through?

$$S=VT=4800 \text{ feet.}$$

Example 18. A power of $P=6500$ effects is operating for a time of $T=12$ seconds with a force $F=240$ pounds. Required the space passed through?

$$S=\frac{PT}{F}=\frac{6500\times12}{240}=325 \text{ feet.}$$

Example 19. To what height can a steam-engine of $\text{HP}=6$ horse-power lift a weight of 25 tons in a time of 5 minutes?

$$F=25\times2240=56000 \text{ pounds.}$$

$$T=5\times60=300 \text{ seconds.}$$

The height $S=\frac{550\times\text{HP}\,T}{F}=\frac{550\times8\times300}{56000}=23.6$ feet.

Example 20. A work of $K=7280$ foot-pounds is to be accomplished by a force of $F=24$ pounds. In what space can the force do the work?

$$S=\frac{K}{F}=\frac{7280}{24}=304 \text{ feet.}$$

Horse-Power.

Example 21. How many horse-power are there in $P = 56680$ effects?

$$\text{HP} = \frac{P}{550} = \frac{56680}{550} = 103 \text{ horse-power.}$$

Example 22. A weight of three tons is to be raised with a velocity of $V = 6$ feet per second. Required the horse-power?

$$\text{HP} = \frac{F\,V}{550} = \frac{3 \times 2240 \times 6}{550} = 73.3 \text{ horse-power.}$$

Example 23. A steam-crane is to be constructed to lift 30 tons 12 feet high in 5 minutes. Required the horse-power?

Force $F = 30 \times 2240 = 67200$ pounds.

Time $T = 5 \times 60 = 300$ seconds.

$$\text{HP} = \frac{F\,S}{550\,T} = \frac{67200 \times 12}{550 \times 300} = 5 \text{ horse-power, nearly.}$$

Example 24. What horse-power is required to accomplish a work of $K = 346000$ foot-pounds in $T = 5$ seconds?

$$\text{HP} = \frac{K}{550\,T} = \frac{346000}{550 \times 5} = 12.6 \text{ horse-power.}$$

WORK IN FOOT-POUNDS.

Example 25. How much work is accomplished with a force of $F = 280$ pounds, moving with a velocity of $V = 9$ feet per second for a time of $T = 1200$ seconds, or 20 minutes?

$$K = F\,V\,T = 280 \times 9 \times 1200 = 3024000 \text{ foot-pounds,}$$

or 2.1 workmandays.

Example 26. How much work can be accomplished by a power of $P = 36$ effects during $T = 4$ seconds?

$$K = P\,T = 36 \times 4 = 144 \text{ foot-pounds.}$$

Example 27. A weight of 25 tons is lifted $S = 18$ feet. Required the work?

$$K = F\,S = 25 \times 2240 \times 18 = 1008000 \text{ foot-pounds.}$$

Example 28. How much work is accomplished per minute by an engine of $\text{HP} = 48$ horse-power?

$$K = 550\,\text{HP}\,T = 550 \times 48 \times 60 = 1584000 \text{ foot-pounds.}$$

§ 13. CIRCULAR OR ROTARY MOTION.

In this case it is supposed that the force F is applied in the direction of a tangent to the circle of radius r in feet, like that of a belt or rope over a pulley, or in all kinds of gearing.

n = revolutions of the circle per minute.

N = total revolutions in the time T, or for generating a definite circular space S, and also for the accomplishment of a definite work K.

DYNAMICAL FORMULAS FOR CIRCULAR MOTION.

Velocity in the Periphery of the Circle in Feet per Second.

$$V=\frac{2\pi r n}{60}=0.10472\, r n. \qquad 29$$

$$V=\frac{2\pi r N}{T}. \qquad 30$$

Revolutions of the Circle per Minute.

$$n=\frac{60\,V}{2\pi r}=\frac{9.425\,V}{r}. \qquad 31$$

$$n=\frac{5250\text{ HP}}{F r}. \qquad 32$$

Total Revolutions N.

$$N=\frac{S}{2\pi r}. \qquad 33$$

$$N=\frac{K}{F\,2\pi r}. \qquad 34$$

Force in Pounds, acting in the Periphery.

$$F=\frac{5250\text{ HP}}{r n}. \qquad 35$$

$$F=\frac{K}{2\pi r N}. \qquad 36$$

Horse-Power, acting in the Periphery.

$$\text{HP}=\frac{F\,2\pi r n}{550\times 60}=\frac{F r n}{5250}. \qquad 37$$

$$\text{HP}=\frac{F\,2\pi r N}{550\,T}=\frac{F r N}{87.5\,T}. \qquad 38$$

Work in Foot-Pounds, accomplished in N Revolutions, or in the Time T.

$$K=F\,2\pi r N. \qquad 39$$

$$K=\frac{F\,2\pi r n}{60\,T}. \qquad 40$$

EXAMPLES FOR CIRCULAR MOTION CORRESPONDING WITH THE FORMULAS.

Example 29. The radius of a wheel or a crank-pin is $r=2.5$ feet, and makes $n=56$ revolutions per minute. Required the velocity in the circumference?

$$V=\frac{2\pi r n}{60}=0.1472\, r n=0.1472\times 2.5\times 56=20.6 \text{ feet per second.}$$

Example 30. Required the velocity in the periphery of a fly-wheel of radius $r = 8$ feet, and making $N = 125$ revolutions in $T = 164$ seconds?

$$V = \frac{2\,\pi\,r\,N}{T} = \frac{6.28 \times 8 \times 125}{164} = 39.25 \text{ feet per second.}$$

§ 14. DYNAMICS OF STEAM-ENGINES.

The following formulas are for a double-acting steam-engine, of which the stroke of piston $= s$ in feet.

$F =$ force or pressure of steam on the piston.

If the steam is expanded in the cylinder, F means the mean pressure on the whole piston throughout the stroke s.

$n =$ double-strokes per minute.

$\text{IP} =$ horse-power of the engine.

Velocity of the piston in feet per second, $V = \frac{2\,s\,n}{60} = \frac{s\,n}{30}$. . 41

Horse-power of the engine, $\text{IP} = \frac{2\,F\,s\,n}{550 \times 60} = \frac{F\,s\,n}{16500}$. . . . 42

Work done in the time T, $K = \frac{2\,F\,s\,n\,T}{60} = \frac{F\,s\,n\,T}{30}$. . . . 43

Work done in N double-strokes, $K = 2\,F\,s\,N$. . . . 44

Let A denote the area of the steam-piston in square inches, and $p =$ mean steam-pressure in pounds per square inch. Then the force on the piston will be $F = A\,p$. 45

And the horse-power, $\text{IP} = \frac{A\,p\,s\,n}{16500}$. 46

This is the gross horse-power of the engine, including that expended in friction and working the pumps, and is generally called

Indicated Horse-Power.

The indicated horse-power is calculated from the indicator diagram or card taken for that purpose.

§ 15. Load of Burden that can be Carried by Man and Animals.

Carriers.	Road, kind of.	Force. Burden, in pounds.	Velocity. Feet per second.	Time. Hours per day.	Space, in miles.
Man	Good level....	100	3	7	14.3
Man	Ordinary......	95	2.5	7	12
Man	Mountainous.	50	3.5	10	23.8
Llama of Peru	Mountainous.	100	3.5	10	23.8
Donkey	Good level....	300	3.5	10	23.8
Donkey	Mountainous.	200	3.5	10	23.8
Mule	Good level....	500	5.0	10	34
Mule	Mountainous.	400	4.5	10	40.6
Horse...........	Good level....	300	6	8	32.7
Horse...........	Mountainous.	300	4.5	8	24.5
Camel	Deserts	1000	3 to 4	12	30 to 40
Elephant......	Ordinary......	1800	3 to 4	10	35

Man or Animal Working a Machine.

Working-man or animal.	Machine which is worked.	Elements.			Functions.		
		Force, in pounds.	Velocity. Feet per second.	Time. Hours per day.	Space, in feet.	Power, in effect.	Work, in foot-pounds.
		F	V	T	S	P	K
Man......	Rope and pulley	50	0.8	6	17280	40	864800
Man......	Crank................	20	2.5	8	72000	50	1440000
Man......	Tread-wheel*.....	144	0.5	8	14400	72	2073600
Man......	Tread-wheel†.....	30	2.5	8	72000	75	2160000
Man......	Draws or pushes.	30	2	8	57600	60	1728000
Horse....	Horse-mill	106	3	8	64800	318	6768800
Horse....	Horse-mill	72	9	5	162000	648	11664000
Horse....	4-wheel carriage.	154	3	10	108000	462	16632000
Horse..	Revolving mill platform.	100	3	8	86400	300	8640000
Mule ..		66	3	8	64800	198	4276800
Ass.....		33	3	8	64800	99	2138400

* Axis horizontal. † Axis 24° from vertical.

§ 16. HORSE-POWER REQUIRED TO DRIVE DIFFERENT MACHINES.

WATER-WORKS.

	HP
For every 100 gallons of water pumped per minute to a vertical height of 100 feet, requires	5
For every million (1,000,000) gallons pumped per 24 hours to a height of 100 feet, requires	35

ROLLING-MILLS.

	HP
For every square foot of heated iron plate passing through the rollers, requires	5
Bar-iron mills. Two pairs of rough and two pairs of finishing rollers, six puddle furnaces, two welding furnaces, making 10 tons of bar iron per 24 hours, rollers making 70 revolutions per minute, require	80

SAW-MILL—ALTERNATIVE.

For every 100 square feet sawed per hour in dry oak, requires	5
Dry pine, per 100 square feet, requires	3

CIRCULAR-SAWS.

A saw 3 feet in diameter, making 300 revolutions per minute, will saw 50 square feet per hour in dry oak, and requires .	2
Dry spruce, 90 square feet per hour	2

THRESHING-MACHINES.

Velocity of feed-rollers at the circumference, 0.55 feet per second. Diameter of threshing-cylinder, 3.5 feet, and 4.5 feet long, making 300 revolutions per hour, can thresh 30 to 40 bushels of oats, and from 25 to 35 bushels of wheat per hour .	4

FLOUR-MILLS.

One pair of mill-stones 4 feet in diameter, making 130 revolutions per minute, can grind 6 bushels of wheat to fine flour per hour	5
Can grind 6 bushels of rye to coarse flour per hour . .	3
For every 100 pounds of fine flour ground per hour, requires .	1

§ 17. DREDGING-MACHINES.

The accompanying illustrations of dipper and grapple dredges are furnished by the American Dredging Company of Philadelphia.

Fig. 61.

IMPROVED "DIPPER-DREDGE."

Dipper-Dredge.—Fig. 61 represents the ordinary dipper-dredge, consisting of one scoop, worked with a triple chain wound on a 15-inch drum, and driven by a pair of engines 10 inches in diameter by 15 inches stroke of cylinders. Under ordinary work the scoop makes 30 to 40 dips per hour, and takes up about two cubic yards, or three tons, of materials each dip.

The dipper-dredge is used in harbors and docks, and also in railroad excavations.

Fig. 62.

PATENT IMPROVED "GRAPPLE-DREDGE."

Grapple-Dredge.—Fig. 62 represents the grapple-dredge, consisting of a double scoop opening in the bottom like a mouth, and takes up about five tons of materials each grapple. It is worked by a single chain wound on a drum three feet in diameter, with a pair of engines 14 inches diameter by 20 inches stroke of cylinders. Under ordinary work it makes 50 to 60 grapples per hour.

Ladder-Dredge.—The ladder-dredge consists of an endless chain upon which a number of buckets are fixed and work continually like a Noria. This appears to be the best form of dredge for deepening harbors, but is not so well suited for docks, where the dipper and grapple dredges are the best.

FORMULAS FOR THE LADDER-DREDGE.

$$\text{IP} = T\left(\frac{h}{700}+k\right), \qquad T=\frac{700\ \text{IP}}{h+700\,k}, \qquad F=\frac{550\,T\,k}{V}.$$

IP = horse-power required for excavating and raising the materials.
T = tons of materials excavated and raised per hour.
h = height in feet to which the excavated materials are raised.
F = force in pounds required to feed the dredge ahead.
V = velocity of the buckets in feet per second.

$k = 0.1$ for hard clay with gravel.	$k = 0.05$ for common clay and sand.
$k = 0.07$ for hard pure clay.	$k = 0.04$ for soft clay and loose sand.

§ 18. POWER REQUIRED TO PROPEL STEAMBOATS AT DIFFERENT SPEEDS.

M = nautical miles or knots per hour.

T = displacement in tons, which must be well proportioned for speed.

HP = horse-power required to propel the vessel M miles per hour.

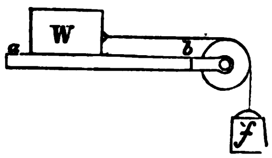
$$\text{HP} = \frac{M^3 \sqrt[3]{T^2}}{228}.$$

Tons.	NAUTICAL MILES OR KNOTS PER HOUR.									
	2	4	6	8	10	12	14	16	18	20
T	HP	HP	HP	HP	HP	HP	HP	HP	HP	HP
1	0.035	0.280	0.794	2.240	4.386	7.792	12.03	17.92	25.60	35.09
2	0.055	0.444	1.260	3.555	6.960	10.08	19.09	28.44	40.63	54.68
3	0.075	0.598	1.651	4.787	9.123	13.21	25.00	38.30	53.25	72.98
4	0.084	0.673	1.900	2.389	11.05	15.20	30.31	43.11	64.51	88.41
5	0.102	0.818	2.207	6.550	12.22	17.65	35.17	52.40	74.85	97.76
6	0.115	0.924	2.620	7.392	14.47	20.96	39.70	59.13	84.48	115.8
7	0.128	1.025	2.906	8.198	16.05	23.25	44.03	65.50	93.70	128.4
8	0.130	1.120	3.176	8.96	17.54	31.17	48.12	68.70	102.4	140.3
9	0.151	1.211	3.430	9.690	19.00	27.44	52.10	77.52	110.1	152.0
10	0.162	1.300	3.684	10.40	20.35	29.47	55.82	83.20	118.8	162.8
15	0.213	1.702	4.827	13.62	26.66	38.62	73.50	108.9	156.0	213.2
20	0.258	2.064	5.845	16 51	32.25	46.76	88.88	132.1	188.5	258.0
30	0.338	2.704	7.675	21.63	42.42	60.40	116.5	173.0	248.0	339.4
40	0.409	3.272	9.280	26.17	51.25	74.24	140.5	209.3	300.0	410.0
50	0.474	3.792	10.79	30.33	59.53	86.32	163.5	242.6	346.2	476.2
60	0.538	4.304	12.20	34.43	67.35	97.60	185.0	285.4	393.5	538.8
70	0.597	4.676	13.55	38.21	74.85	108.4	205.5	305.7	437.0	598.8
80	0.650	5.200	14.8	41.60	81.60	118.4	224.0	332.8	476.0	652.8
90	0.705	5.640	16.00	45.22	88.40	128.0	242.5	361.7	516.0	707.2
100	0.755	6.040	19.40	48.4	94.5	163	259	387	551	756
150	0.990	7.720	22.51	61.76	124.0	180.0	341.5	494.1	724.0	992.0
200	1.200	9.600	32.5	76.9	150	260	412	615	875	1201
300	1.575	10.60	42.4	100	196	340	540	806	1146	1573
400	1.910	15.28	51.4	122	238	412	654	976	1402	1907
500	2.213	17.70	59.6	141	276	478	759	1131	1611	2213
600	2.50	20.00	67.2	160	313	540	856	1280	1820	2500
700	2.780	22.24	74.6	177	377	599	938	1417	2016	2770
800	3.025	24.20	81.5	194	388	654	1038	1548	2206	3026
1000	3.500	28.00	94.6	225	439	759	1206	1798	2560	3514
1500	4.611	36.88	124	295	575	995	1580	2355	3352	4605
2000	5.600	44.80	150	356	696	1205	1913	2854	4060	5570
3000	7.350	58.80	197	467	913	1582	2508	3740	5318	7300
4000	8.87	70.96	238	567	1105	1912	3038	4530	6444	8847
6000	11.44	91.52	303	742	1448	2507	3981	5935	8446	11586

§ 19. FRICTION.

Sliding Friction is the force required to rub or slide one surface upon another. For the same kind of surfaces the force of friction is proportionate to the pressure of contact, and independent of the velocity with which the rubbing or sliding body moves. Within certain limits the friction is also independent of the extent of surface in contact.

Fig. 63.



Let $a\ b$ represent a horizontal surface on which is placed a body W in close contact with $a\ b$. The body W is attached to a weight f by a rope over a pulley. Adjust the weight f so that it will barely move the body W; then f is the force of friction of the surfaces in contact. If the body W be started with a certain velocity V, the weight f will continue that velocity uniformly; but if it is greater than the friction, the velocity will be accelerated.

W = weight of the body in pounds, which is the force or pressure of contact of the friction surfaces.
f = force in pounds of the friction.
Φ = ratio of W and f, or the coefficient of friction.
V = velocity in feet per second of the motion.
S = space of motion in feet.

$$\Phi = \frac{f}{W}, \quad W = \frac{f}{\Phi}, \quad \text{and } f = W\Phi.$$

Friction power, $P = f\,V.$

Friction horse-power, $HP = \dfrac{f\,V}{550}.$

Work of friction, $K = f\,S.$

The work expended on friction is generally converted into heat, which is not utilized, but lost. It is therefore of great importance in the working of machinery to reduce the work of friction to the lowest possible amount, for which reason lubricating substances are introduced between the friction surfaces, such as powdered graphite or soapstone, and all kinds of fatty substances, such as oil, tallow, lard, soap, etc.; all of which reduce the friction coefficient, but to a different degree, depending on how and on what kind of surfaces the lubrication is used.

The force of friction can be ascertained only by experiments which have been made by Coulomb, Vince, G. Rennie, N. Wood; and the most complete and reliable experiments on friction were made by Arthur Morin in the years 1831, '32, and '33, at the expense of the French government.

Friction surfaces must be either plane, cylindrical, spherical, conical or any figure concentric with an axis of rotation.

§ 20. PLANE FRICTION SURFACE.

The friction coefficient for plane surfaces is best determined by a body sliding on an inclined plane.

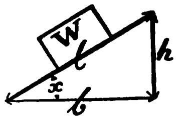
h = height of the inclined plane.

l = length, and b = base.

W = weight of the sliding body.

f = force of friction, which is equal to the force of gravity acting to draw the body down on the inclined plane.

Fig. 64.



The inclined plane is so elevated or adjusted that the body barely moves by its own weight.

Let x denote the angle of the inclined plane with the horizon, then the friction coefficient will be

tan.

$$\Phi = \cancel{\sin.}\,x.$$

From the law of statics we have

$$W : f = l : h, \quad \text{and} \quad W h = f l.$$

$$W = \frac{f\,l}{h}, \quad f = \frac{W h}{l}, \quad h = \frac{f\,l}{W}, \quad l = \frac{W h}{f}.$$

tan.

$$\text{Friction coefficient } \Phi = \cancel{\sin.}\,x = \frac{h}{b}$$

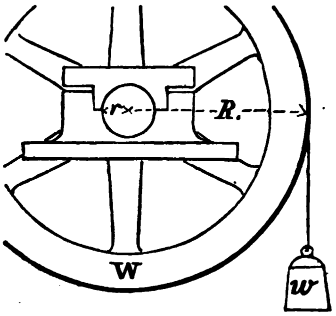
The friction coefficient is independent of the extent of areas in contact until near the point of abrasion.

When the Great Eastern was to be launched it was found that she could not be pulled down on her ways with the greatest force that could be applied; whereupon Mr. Brunell invited George Stephenson to come to the launch and give advice on the same. Upon his arrival, George Stephenson remarked that the weight of the ship had exceeded the abrasion on the ways, and that the ship could not be launched without increasing the area of the ways; which was accordingly done, and the ship went off with considerable ease.

§ 21. CYLINDRICAL FRICTION SURFACE.

Axles and shafts, bearing in journals, are generally cylindrical.

Fig. 65.



W = weight or pressure in the journal.

w = weight or force applied to give rotation or motion.

R = radius upon which the force w acts, in feet.

r = radius in feet of the journal, upon which the force of friction f acts.

n = number of revolutions per minute.

$R : r = f : w$ and $Rw = fr$.

$$R = \frac{fr}{w}, \quad r = \frac{Rw}{f}, \quad f = \frac{Rw}{r}, \quad w = \frac{fr}{R}.$$

Friction coefficient $\Phi = \dfrac{f}{W} = \dfrac{Rw}{Wr}$.

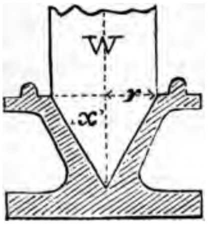
Horse-power of friction $HP = \dfrac{2\pi rnf}{33000} = \dfrac{2\pi rnW\Phi}{33000}$.

$$HP = \frac{rnf}{5252} = \frac{rnW\Phi}{5252}.$$

§ 22. CONICAL FRICTION SURFACE.

This illustration represents a vertical shaft supported on a conical pivot.

Fig. 66.



W = force or pressure in pounds in the direction of the centre line of the shaft.

r = radius of the base of the cone in feet.

x = half the angle of the cone.

f = force in pounds of the friction acting on the mean radius, 0.707 r.

w = pressure in pounds on the conical friction surface.

n = number of revolutions per minute.

$w = W \operatorname{cosec} x$.

$f = w\,\Phi$.

In this case the friction coefficient is the same as for cylindrical journals.

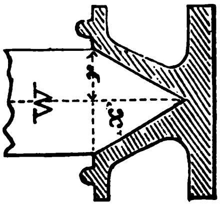
Friction horse-power $HP = \frac{0.70711\, r\, n\, w\, \Phi}{5252.1} = \frac{r\, n\, w\, \Phi}{7427.5}$.

For a flat circular surface $x = 90°$ and $cosec.x = 1$.

Friction horse-power $HP = \frac{r\, n\, W \Phi}{7427.5}$.

§ 23. HORIZONTAL SHAFT.

Fig. 67.



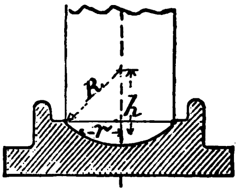
When the shaft is horizontal and the pressure vertical, the pressure on the conical friction surface will be $w = W\, sec.x$.

$$HP = \frac{r\, n\, w\, \Phi}{7427.5}.$$

§ 24. SPHERICAL FRICTION SURFACE.

For a spherical pivot the mean radius is at half the height of the segment.

Fig. 68.



R = radius of the sphere.
r = mean radius of friction.
W = pressure on the friction surface.

$$HP = \frac{r\, n\, W \Phi}{5252}.$$

The friction coefficient in this case will be the same as that for cylindrical journals.

8

TABLE I.

§ 25. **SUMMARY OF MORIN'S EXPERIMENTS ON FRICTION.**

= fibres moving parallel on one another.
+ fibres moving at right angle to fibres.
± end of fibres moving parallel to fibres.
⊥ end of fibres moving at right angle to fibres.
I fibres moving end to end.

Materials. Moving.	Materials. Stationary.	Condit'n of fibres.	Lubrication.	In motion. Coef. Φ.	In motion. Angle x.	Starting. Coef. Φ.	Starting. Angle x.
Oak	Oak	=	non	0.48	25° 39′	0.62	31° 48′
Oak	Oak	=	Dry soap	0.16	9 6	0.44	23 45
Oak	Oak	I	non	0.34	18 47	0.54	28 22
Oak	Oak	I	Water	0.25	14 3	0.71	35 23
Oak	Oak	±	non	0.19	10 46	0.43	23 16
Elm	Oak	=	non	0.43	23 17	0.38	20 49
Elm	Oak	I	non	0.45	24 14	0.57	29 41
Elm	Oak	±	non	0.25	14 3		
Ash, fir	Oak	=	non	0.38	20 48	0.53	27 56
Iron	Oak	=	non	0.62	31 48	0.62	31 48
Iron	Oak	=	Water	0.26	14 35	0.65	33 2
Iron	Oak	=	Dry soap	0.21	11 52		
Cast-iron	Oak	=	Water	0.22	12 25	0.65	33 2
Cast-iron	Oak	=	Dry soap	0.19	10 46		
Copper	Oak	=	non	0.62	31 48	0.62	31 48
Iron	Elm	=	non	0.25	14 3		
Cast-iron	Elm	=	non	0.20	11 19		
Leather	Oak	=	non	0.27	15 7		
Leather	Cast-	⊥	non	0.56	25 15		
Leather	iron	±	Water	0.36	19 48		
Leather	or	±	Wat. oil	0.23	12 58		
Leather	Brass	±	Oil	0.15	8 32		
Hemp	Oak	=	non	0.52	27 29	0.80	38 40
Hemp	Oak	I	Water	0.33	18 16	0.87	41 2
Iron	Iron	=	non	0.44	23 45		
Iron	Brass	=	non	0.18	10 13		
Cast-iron	Brass	"	non	0.15	8 32		
Brass	Brass	"	non	0.20	11 19		
Brass	Cast-iron	"	non	0.22	12 25		
Brass	Iron	"	non	0.16	9 6		
Hard	Hard	"	Well	0.07 to	4 1	0.10 to	5 43
Wood or	Wood or	"	greased	0.08	4 35	0.15	8 32
Iron	Iron	"	slightly	0.15	8 32		

TABLE II.

§ 26. Friction in Shaft-journals.—Morin.

Materials in Shaft.	Materials in Journal-box.	Lubricating substances.	Coefficient ϕ when greased with oil perpet.	Coefficient ϕ when greased with oil at intervals.
Cast-iron........	Cast-iron........	Coated with grease........	0.054	0.07 to 0.08
Cast-iron........	Cast-iron........	Oil and water...............	0.28	0.08
Cast-iron........	Cast-iron........	Coated with asphaltum..	0.19	0.054
Cast-iron........	Lignumvitæ ...	Oil or hog's lard...........	......	0.10
Wrought-iron..	Cast-iron........	Coated with grease........	0.054	0.07 to 0.08
Wrought-iron..	Brass............	Coated with gum.	......	0.09
Wrought-iron..	Brass............	Greasy and wet............	......	0.19
Wrought-iron..	Brass............	Scarcely greasy............	......	0.25
Brass............	Brass............	Coated with oil............	......	0.10
Brass............	Brass............	With hog's lard...	......	0.09
Brass............	Cast-iron........	Oil or tallow................	......	0.048
Lignumvitæ....	Lignumvitæ....	Hog's lard...................	......	0.04

TABLE III.

§ 27. Friction Coefficient by Different Observers.

Materials. Moving.	Materials. Stationary.	Condition of surface.	Coefficient ϕ.	Angle limit.
Soft calcareous stone...	Soft cal. stone.....	Well-dressed...	0.74	36° 30′
Hard calcareous stone..	Hard cal. stone...	Well-dressed...	0.75	36 52
Common brick...........	Common brick....	Common.........	0.67	33 50
Soft calcareous stone...	Soft cal. stone......	Fresh mortar...	0.74	36 30
Freestone................	Freestone..........	Smooth..........	0.71	35 23
Freestone................	Freestone..........	Fresh mortar...	0.66	33 26
Hard calcareous stone..	Hard cal. stone...	Polished.........	0.58	30 7
Calcareous stone.........	Calcareous stone..	Rough...........	0.78	37 58
Wood-box................	Pavements	Common.........	0.58	30 7
Wood-box................	Beaten earth.......	Common.........	0.33	18 16
Libage stone..............	Bed of clay........	Dry	0.51	27 2
Libage stone..............	Bed of clay........	Damp........ ...	0.34	18 47
Oak	Calcareous stone..	Dry...............	0.36	32 13

TABLE IV.

§ 28. Friction Coefficients for Different Pressures up to the Limit of Abrasion.

FROM EXPERIMENTS BY MR. G. RENNIE.

("Philosophical Transactions," 1829.)

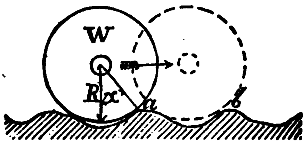
Pressure per square inch.	Wrought-iron upon wrought-iron.	Wrought-iron upon cast-iron.	Steel upon cast-iron.	Brass upon cast-iron.
32.5	0.140	0.174	0.166	0.157
187	0.250	0.275	0.300	0.255
240	0.271	0.292	0.233	0.219
277	0.285	0.320	0.340	0.214
315	0.297	0.329	0.344	0.211
336	0.312	0.333	0.347	0.215
373	0.350	0.351	0.351	0.206
411	0.376	0.353	0.353	0.205
448	0.395	0.365	0.354	0.208
485	0.403	0.366	0.356	0.221
523	0.409	0.366	0.357	0.223
560	Abrasion.	0.367	0.358	0.233
597		0.367	0.359	0.234
635		0.367	0.367	0.235
672		0.376	0.403	0.233
709		0.434	Abrasion.	0.234
747		Abrasion.		0.235
784				0.232
821				0.273

§ 29. ROLLING FRICTION.

Rolling-friction is the resistance of uneven surfaces rolling on one another, like that of a wheel rolling on a road. The coefficient of rolling friction represents the unevenness of the surfaces in contact, for if these surfaces were perfectly hard and smooth, there should be no rolling-friction.

Smooth Wheel on Irregular Hard Road.

Fig. 69.



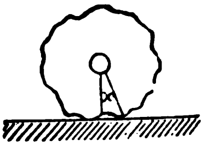
A wheel of radius R and weight W rolling on an uneven road, and strikes a projection at a, which presents a resistance f to the forward motion.

$$f = W \tan.x.$$

This resistance will be diminished to nothing when the centre of the wheel is vertical above *a*, after which a similar force acts *with* the motion until the wheel strikes the second projection *b*. The forces of the projection thus act alternately against and with the motion, so that there would be no force lost; but when the wheel strikes the projection, work is performed in crushing or wearing the surfaces in contact; and it is the force of this work which makes the rolling-friction.

Irregular Wheel on a Smooth Hard Road.

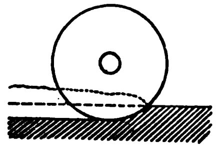
Fig. 70.



The wheel may also be irregular and run on a smooth road, or both the wheel and road may be irregular. In either case the rolling-friction is the force of the work expended in wearing the surface in contact.

On Soft or Muddy Roads.

Fig. 71.



When a wagon is run on a soft or muddy road the path of the wheels represents the work done by the rolling-friction.

This rolling-friction can be measured on an inclined muddy road by loading the wagon until it will barely move by its own weight.

X = angle of inclination of the road.

W = weight of the load and wagon.

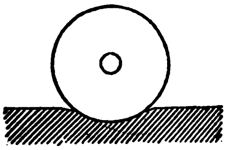
f = friction resistance in the mud, omitting the rotary friction in the axis.

$$f = W \sin.x = W\Phi.$$

$$\text{Rolling-friction coefficient } \Phi = \frac{f}{W} = \sin.x.$$

Hard Wheel on Elastic Road.

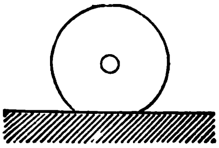
Fig. 72.



A hard wheel running on a perfectly elastic road will leave no path behind, and there is thus no work performed by rolling-friction, because the forces of the elastic road act equally for and against the motion.

Elastic Wheel on Hard Road.

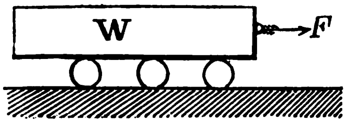
Fig. 73.



In this case the wheel will be compressed on the road, and the forces acting with and against the motion will be alike, and there will be no rolling-friction if the wheel is perfectly elastic.

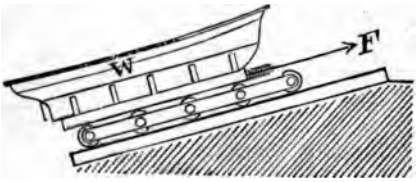
A Load on Rollers.

Fig. 74.



A load moved on rollers placed loose on the road will move twice as fast as the rollers; and the roller left behind is placed in front of the load continually until the rolling distance is completed. Any number of rollers may be used; and sometimes they are connected by rods, like in slip railways for hauling up ships. The ship will move twice as fast as the roll-carriage. The length of the ship is L, and the distance to be hauled is D; the length l of the roll-carriage must be

Fig. 75.



$$l = L + \tfrac{1}{2}D.$$

W = weight of the ship and carriage.
x = angle of inclination of the railway.
F = hauling force.

$F = W \sin.x.$, omitting rolling-friction, which may be about 5 per cent. of W.

It requires nearly double the force to start the motion that is to continue the same.

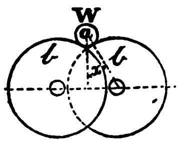
On hard roads the rolling-friction is proportionate to the load and inversely as the diameter of the wheels, but is independent of the width and number of wheels.

On soft or muddy roads the rolling-friction increases slightly with the load and diminishes nearly as the width of the wheel.

§ 30. CYLINDER ROLLING-FRICTION.

The rolling-friction of smooth surfaces in contact is very small, which circumstance is sometimes utilized by running shafts on cylinders or rollers, as represented by Fig. 76. The shaft a runs on the rollers b, b.

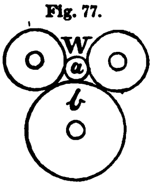
Fig. 76.



W = pressure of the shaft on the rollers.
R = radius of the rollers.
r = radius of the roller-journals.
r' = radius of the shaft.
Φ = coefficient of friction in roller-journals.
x = angle as shown on the illustration.
f = force of friction on the shaft radius r'.

$$f = \frac{W\Phi r}{R\,sec.x}.$$

Fig. 77.



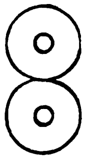
The weight of the shaft a may also be supported by one roller b, as represented by Fig. 77, in which case

$$f = \frac{W\Phi r}{R}.$$

FRICTION-GEAR.

For light work motion can be transmitted from one shaft to another by forcible contact of two wheels, Fig. 78.

Fig. 78.



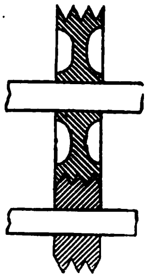
W = pressure of contact.

Φ = coefficient for sliding-friction.

f = tangential force transmitted on the wheels.

$$f = W\Phi.$$

Fig. 79.



GROOVED FRICTION-GEAR.

The tangential force of friction-gear can be increased by grooving the peripheries of the wheels, as shown by the illustration.

x = angle of the groove.

$$f = W\Phi\, cosec.\tfrac{1}{2}x.$$

§ 31. ROLLING-MILL.

The rolling-friction in a rolling-mill is the force of the work in compressing the body passing between the rollers. By knowing the motive-power and weight of the fly-wheel, the rolling work and friction can be determined by their performance.

Assume the mill to be worked by a steam-engine attached direct to the fly-wheel shaft and roller, and

F = force of steam-pressure on the piston.

S = length of stroke.

W = weight of the fly-wheel.

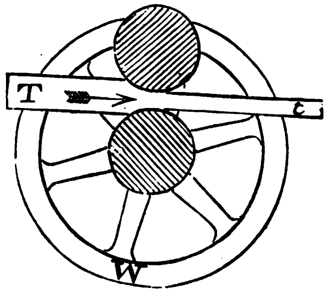
X = radius of gyration of fly-wheel.

n = number of revolutions per minute when the body enters the rollers.

n' = revolutions when the body leaves the rollers.

N = revolutions in which the body was rolled.

Fig. 80.



The work of the steam-engine in rolling the object will then be

$$k = 2\,FSN \text{ in foot-pounds.} \quad . \quad . \quad . \quad 1$$

The work done by the fly-wheel in the same operation will be

$$k = \frac{WX^2}{5867}(n^2 - n'^2). \quad . \quad . \quad . \quad . \quad 2$$

The whole work in rolling the body will then be

$$K = \frac{WX^2}{5867}(n^2 - n'^2) + 2\,FSN. \quad . \quad . \quad . \quad 3$$

T = thickness, and l = length of the body when entering, and
t = thickness, and L = length when leaving the rolls.
f = force of compression.
d = length of the object pressed by the rolls.
The work of compression will be

$$K = \frac{f}{2\,d}(L + l)(T - t), \quad . \quad . \quad . \quad . \quad 4$$

which should be equal to the work expended by the engine and fly-wheel.

$$\text{Force } f = \frac{2\,K\,d}{(L + l)(T - t)}. \quad . \quad . \quad . \quad . \quad 5$$

The work K must be calculated by Formula 3, and inserted for K in Formula 5, for finding the force of compression of the rollers.
All linear dimensions are expressed in feet and forces in pounds.

§ 32. TRACTION ON LEVEL ROADS.

Traction on roads is the force required to pull or move a load on a horizontal road, and which includes both rolling- and axle-frictions. On very smooth and hard roads, like that of a train on steel-rails in good order, the rolling-friction is a small item of the axle-friction.

W = weight of the load moved on the road.
F = force of traction.
f = traction coefficient.

$$f = \frac{F}{W}, \qquad F = Wf, \text{ and } W = \frac{F}{f}.$$

The following table gives the traction coefficient on different kinds of horizontal roads:

TABLE V.

Traction Coefficients.

<table>
<tr><th colspan="2">Carriages.</th><th>Roads.</th><th>Traction f.</th></tr>
<tr><td colspan="2">Railway trains</td><td>New steel rails</td><td>0.0025</td></tr>
<tr><td colspan="2">Railway trains</td><td>Iron rails, good condition</td><td>0.0030</td></tr>
<tr><td colspan="2">Railway trains</td><td>Worn iron rails</td><td>0.0040</td></tr>
<tr><td colspan="2">Wagons</td><td>Smooth stone pavement</td><td>0.0048</td></tr>
<tr><td colspan="2">Wagons</td><td>Good street pavement</td><td>0.0080</td></tr>
<tr><td colspan="2">Wagons</td><td>Turnpikes</td><td>0.0120</td></tr>
<tr><td colspan="2">Wagons</td><td>Coarse gravel</td><td>0.020</td></tr>
<tr><td colspan="2">Wagons</td><td>Common bad roads</td><td>0.060</td></tr>
<tr><td colspan="2">Wagons</td><td>Loose ground</td><td>0.200</td></tr>
<tr><td rowspan="9">Morin's experiments.</td><td>Artillery wagons</td><td>Good dry sand</td><td>0.0260</td></tr>
<tr><td>Artillery wagons</td><td>Wet sand</td><td>0.0250</td></tr>
<tr><td>Cart without springs</td><td>Hard road</td><td>0.0195</td></tr>
<tr><td>Cart without springs</td><td>Moist gravel</td><td>0.0210</td></tr>
<tr><td>Cart without springs</td><td>Dry gravel</td><td>0.0200</td></tr>
<tr><td>Cart with springs</td><td>Ordinary road</td><td>0.0245</td></tr>
<tr><td>Cart with springs</td><td>Muddy road</td><td>0.0272</td></tr>
<tr><td>Carriage with 6 wheels</td><td>Sand, rutted</td><td>0.0463</td></tr>
<tr><td>Two carriages with 12 wheels.</td><td>Muddy road</td><td>0.0476</td></tr>
</table>

§ 33. TRACTION ON INCLINED ROADS.

On inclined roads the weight of the load acts against or with the traction as the direction of motion is up or down the incline.

x = angle of inclination of the road with the horizon.

F = tractive force up or down the inclined road.

$$F = Wf\,cos.x \pm W\,sin.x.$$
$$F = W(f\,cos.x \pm sin.x).$$

+ when the motion is upward. − when the motion is downward.

The road may be so inclined that no tractive force is required to move the load down, which will happen when

$$f\,cos.x < sin.x.$$

The load is in equilibrium on the road when $f\,cos.x = sin.x$, in which case it requires a force of

$$F = W\,sin.x \text{ to draw the load up.}$$

When $sin.x. > f\, cos.x$ it requires a force of

$$F = (sin.x - f\, cos.x)\, W$$

to hold the load or to prevent it from moving down the incline by its own weight.

§ 34. ADHESION ON HORIZONTAL RAILS.

Adhesion on rails is the force of sliding friction of the locomotive wheels. This force of adhesion is equal to the weight of the locomotive multiplied by the friction coefficient in Table I.

w = weight of the locomotive in pounds.
A = force of adhesion, $A = w\, \Phi$.

This force must be greater than the tractive force required to move the train, in order to enable the locomotive to go ahead without sliding the wheels.

W = weight of the train in pounds.
f = coefficient of traction, Table V.

There must be $w\, \Phi > Wf$ to produce motion.
The locomotive wheels will slide on the rails when $Wf > w\, \Phi$.

TABLE VI.

Coefficient of Adhesion on Rails.

Condition of Rails.	Φ
Maximum dryness	0.301
Very dry	0.224
Under ordinary circumstances	0.20
In wet weather	0.141
With snow or frost	0.100

ADHESION ON INCLINED RAILS.

The inclination of the track diminishes the force of adhesion as the *cosine* for the angle. x = angle of inclination of the track.

$$\text{Adhesion, } A = w\, \Phi\, cos.x.$$

The force of traction of the locomotive is limited to the force of

adhesion, and when a train is running up or down an inclined track the tractive force must be

$$F = Wf\,cos.x \pm W\,sin.x = \text{or} < w\,\Phi\,cos.x.$$

+ when the traction is upward.
− when the traction is downward.

The traction and gravity of the train is in equilibrium when

$$Wf\,cos.x = W\,sin.x,$$

or when

$$f\,cos.x = sin.x.$$

The train will run down the track by its own force of gravity, when $sin.x > f\,cos.x$.

The resistance of wind to the train is not included in the formulas for force of traction.

Experiments have shown that the force of traction increases slightly with the velocity; which is mostly due to resistance of the air.

TRACTION-POWER.

The tractive power is equal to the force of traction F multiplied by the velocity in feet per second; and if desired in horse-power divide the product by 550.

When the velocity is expressed in miles M per hour, the horse-power will be

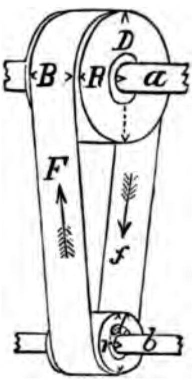
$$HP = \frac{5280\,M\,F}{550} = 9.6\,F\,M.$$

§ 35. BELT AND PULLEYS.

The best and simplest mode of transmitting motion from one shaft to another is by a belt and pulleys, which is very extensively used and it gives the smoothest motion. The motion is transmitted by the frictional adhesion between the surfaces in contact of the belt and pulleys, for which reason that friction must be greater than the tension of the belt, otherwise the belt will slip and fail to transmit all the motion due from the driving pulley. There is always some slip in belt and pulleys, for which reason that mode of transmission is not positive or exact, and cannot be used where precise motions are required.

Fig. 81 represents a belt transmitting motion between two parallel shafts a and b. If the motion is transmitted from a to b, the pulley D is called the driving pulley, and d the driven pulley. The diameters of the pulleys can be of any desired proportions to suit the work of the machine.

Fig. 81.



D = diameter and R = radius in inches of the driving pulley.

d = diameter and r = radius of the driven pulley.

L = length and B = breadth of the belt in inches.

F = force of tension in pounds of the pulling side of the belt.

f = force of tension on the slack side.

V = velocity of the belt in feet per second.

S = distance in inches between the centres of the two pulleys.

N and n = numbers of revolutions per minute of the respective pulleys D and d.

φ = angle in degrees occupied by the belt on the small pulley.

HP = horse-power transmitted by the belt.

Revolutions $N : n = d : D$, diameters. The revolutions are inverse as the diameters.

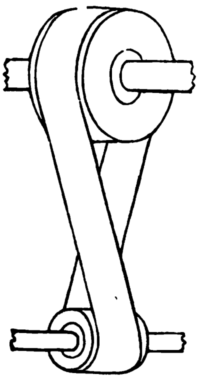
$$N = \frac{n\,d}{D}, \qquad n = \frac{N\,D}{d}, \qquad d = \frac{N\,D}{n}, \qquad D = \frac{d\,n}{N}.$$

The force bearing in the journals of each shaft is $F + f$, or the sum of the tensions of each side of the belt.

The force which transmits the motion is $F - f$, or the difference between the two tensions.

The effective power transmitted is equal to the product of the transmitting force and the velocity, and this power divided by 550 gives the horse-power.

Fig. 82.



The shafts connected by belt and pulleys need not be parallel with each other, but they must lay in parallel planes, as represented by the illustration, Fig. 82. The pulleys must be placed on the shafts so that the driving side of the belt forms right angles with them. The belt can be put on so as to drive the driven pulley in any desired direction.

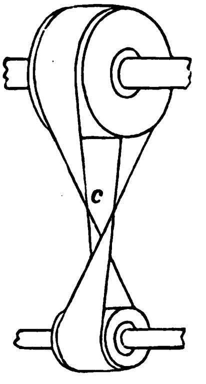
The length L of the belt will be found by the following formula:

$$L = \pi (R + r) + 2\sqrt{S^2 + (R - r)^2}.$$

When the diameters of the pulleys are alike, or $D = d$, the length of the belt will be

$$L = \pi D + 2S.$$

Fig. 83.



The shafts or the belt can be twisted to any desired angle, like in this illustration, Fig. 83, the belt is twisted 180°, and the driven pulley will then run in an opposite direction to that in Fig. 81. The belt should be laid on so as to have the same side on both pulleys, and the insides will then rub flat against one another in the crossing c.

Slip of Belt.

The slip of belt on equal pulleys has been found by experience to vary between 2 and 3 per cent. under ordinary circumstances.

n = theoretical revolutions per minute of the driven pulley.
n' = actual revolution after the slip is deducted.

$$n' = \frac{n\,\varphi}{182}.$$

§ 36. FORMULAS FOR LEATHER BELTS ON CAST-IRON PULLEYS.

For Notation of Letters, see page 96.

Force and Power of Transmission.

$$F-f=\frac{126500\,\text{HP}}{d\,n}. \qquad 1$$

$$F-f=\frac{126500\,\text{HP}}{D\,N}. \qquad 2$$

$$F-f=\frac{550\,\text{HP}}{V}. \qquad 3$$

$$\text{HP}=\frac{V(F-f)}{550}. \qquad 4$$

$$\text{HP}=\frac{d\,n\,(F-f)}{126500}. \qquad 5$$

$$\text{HP}=\frac{D\,N(F-f)}{126500}. \qquad 6$$

$$V=\frac{d\,n}{230}=\frac{D\,N}{230}. \qquad 7$$

$$n=\frac{126500\,\text{HP}}{d\,(F-f)}. \qquad 8$$

Breadth of Belt from Experiments.

$$B=\frac{2.4\,\text{HP}}{n\,d\,\varphi}. \qquad 9$$

$$B=\frac{2.4\,F}{d}. \qquad 10$$

$$B=\frac{432\,F}{d\,\varphi}. \qquad 11$$

$$B=\frac{4320\,\text{HP}}{n\,d}. \qquad 12$$

$$B=\frac{7.8\,F\,V}{n\,d}. \qquad 13$$

$$\text{HP}=\frac{B\,n\,d.}{4320} \qquad 14$$

$$\text{HP}=\frac{B\,n\,d}{24}. \qquad 15$$

$$F=\frac{B\,d}{2.4}. \qquad 16$$

§ 37. VULCANIZED RUBBER BELTS.

The vulcanized rubber belting made by the New York Belting and Packing Company is composed of heavy cotton duck, woven expressly for that purpose and vulcanized between layers of a metallic alloy, by which process the stretch is entirely taken out and the surface made perfectly smooth.

This belting is said to be superior to, and is furnished for about half the price of, that of leather. It will stand a heat of 300° Fahr., and the severest cold will not affect its good quality, even if run in wet places or exposed to damp weather.

The friction of the vulcanized rubber belting is about double that of leather, which makes it less liable to slip on the pulley.

§ 38. MATTER.

Matter is that of which bodies are composed, and occupies space. Matter is recognized as substance in contradistinction from geometrical quantities and physical phenomena, such as color, shadow, light, heat, electricity and magnetism.

We have no knowledge of the origin or source of matter, but only know its existence and obedience to forces. Chemistry has, thus far, dissolved matter into some sixty-five distinct elements, but in the philosophy of mechanics we treat matter only as one simple element in relation to the three physical elements—*force*, *motion* and *time*.

These four elements—force F, motion V, time T and mass M—are what constitute nature, and their different combinations cause the phenomena which we study and observe.

The three first elements—F, V and T—are what constitute life, which physical combination with matter constitutes organic bodies.

Physics divides matter into *atoms*, *molecules*, *particles* and *bodies*.

Atom is the ultimate portion into which matter can be divided.

Molecule is a group of atoms.

Particle is a group of molecules.

Body is a group of particles, consisting of molecules and atoms of matter.

§ 39. ATTRACTION OR GRAVITATION.

It is a well-known and established fact that all bodies in nature have a mutual tendency to attract each other, the action of which is called *universal attraction or gravitation*. It is a constant action between all kinds of matter, which cannot be disturbed by any other cause.

Attraction, gravitation and gravity mean the same physical action, but there exist different kinds of tendencies between different kinds of matter to attract each other, which are independent of the general law of gravitation; such as cohesion, capillar, molecular, chemical affinity, electric and magnetic attractions and repulsions, which are not called gravitation, and which actions are much under the control of human skill. The force of cohesion can be destroyed by a superior force crushing or tearing the body to pieces, or by the application of heat.

The atoms of matter are acted upon by two opposite forces—namely, molecular attraction and repulsion—which cause bodies to exist in three aggregate forms—namely, solid, liquid and gaseous, according to the relation between the two forces.

When the force of attraction is superior to that of repulsion, the body will have the consistency of a solid; but when the two forces are in equilibrium, the body will have a liquid form; and when the force of repulsion is superior to that of attraction, the body will consist in the form of a gas.

It has been anticipated that matter may be converted into a fourth aggregate form—namely, that of an imponderable substance—but the suggestion is not realized as a fact.

All bodies in nature may be, or are capable of being, converted alternately into the three aggregate forms—namely, solid, liquid and gaseous—although we have not yet so succeeded with some bodies. Ice, water and steam are the three aggregate forms of one body.

It appears that the temperature of heat is the force of repulsion, and that the absence of heat (cold) allows the force of attraction to draw the atoms of matter in closer contact. In the case of liquids, it is said that the two forces are in equilibrium, but we know that liquids differ widely in temperature, like that of water and molten iron.

If the force of repulsion increases in some ratio with the temperature, the force of attraction of the atoms of different elements must differ in accordance with the force of repulsion, for, otherwise, liquids could not vary so much in temperature.

The physical constitution of the force of attraction is a mystery to us, and we have yet no hope of its ever being revealed. If there were eye-bolts in each atom of matter, and all were thus connected by elastic strings whose elasticity diminished as the square of their length, some conception could be formed of the nature of attraction; but as there are no such eye-bolts, how does the force of attraction take hold of the atom?

It is self-evident, however, that something must exist between matter to form the connection of attraction; and whether this "something" be elastic strings or not, it cannot be cut off or interfered with in the least by any intervening means.

Universal attraction, means the general force of attraction between the heavenly bodies.

Gravitation, means the same universal force of attraction, but implies that action on or near the surface of the earth or of any other heavenly body.

§ 40. LAW OF ATTRACTION.

The law of universal attraction was anticipated by Copernicus, Tycho Brahe, Kepler, Fermat, Roberval and Hook, and finally established by Sir Isaac Newton. It is expressed as follows:

The force of attraction is directly as the mass, and inversely as the square of the distance.

This expression will hold good when the mass of either one of the attracting bodies is taken as a unit, but, more correctly, the law ought to be expressed thus:

The force of attraction between any two bodies is equal to the mass of the one body, multiplied by that of the other, and the product divided by the square of their distance apart.

This is the universal law of attraction or gravitation upon which our existence wholly depends.

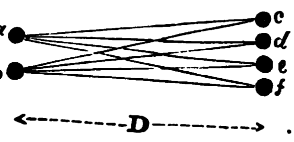
§ 41. ILLUSTRATION OF THE LAW OF ATTRACTION.

Let *a* and *b*, Fig. 84, represent two particles of matter, supposed to constitute one body, and *c*, *d*, *e* and *f* four particles, constituting another body. Each of the six particles is supposed to contain one unit of matter, and the distance *D* between the two bodies to be one unit of length.

Draw straight lines between the particles, as shown in the illustration.

The particle *a* will attract the particle *c*, as well as *d*, *e* and *f*, each with one unit of force; the attraction, therefore, between the particle *a* and the body *c d e f* will be four units of force; and the particle *b* will also attract the body *c d e f* with four units of force; so that the attraction between the bodies *a b* and *c d e f* will be eight units of force, as represented by the eight lines drawn between them.

Fig. 84.



The mass of the body *a b* is 2, that of *c d e f* is 4, and the product of 2 and 4 is 8, the force of attraction, according to Newton's law. It is assumed in the illustration that the distance between the bodies is one unit of length, but if the distance be two units of length, the force of attraction will be only 2, and if the distance *D* is only half a unit of length, the force of attraction will be 32.

Let F denote the force of attraction between any two bodies of masses M and m, D = distance between the bodies, then

$$F = \frac{M m}{D^2}.$$

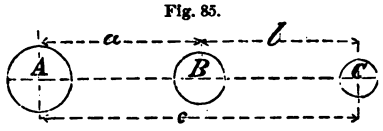
All force, power and work are derived from this law.

The attraction of the sun draws heavenly bodies into it, and the heat generated by the collision is returned into space. It is the heat and light from the sun which decompose carbonic acid in our atmosphere, and promote the growth of vegetation on the surface of the earth, by which we are supplied with food and fuel for motive-power. The burning of the fuel reproduces carbonic acid, which rises into the air, where it is again decomposed by the heat of the sun. Thus, the work of heat is absorbed and reproduced alternately for ever and ever.

These ideas of the sources of work agree with those of George Stephenson, Sir William Thomson, Waterston and Rankin.

§ 42. Suppose three bodies A, B and C, to be fixed in a straight line, and their distances apart as represented by the letters a, b and c.

Fig. 85.



Let the mass or the real quantity of matter in each body be represented by its letter, say $A = 16$, $B = 8$, and $C = 4$, their distances apart being $a = 1$, $b = 2$, and $c = 3$.

Then the forces of attraction between the bodies will be as follows:

Between A and B, force of attraction $= \frac{A\ B}{a^2} = \frac{16 \times 8}{1} = 128.$

Between B and C, force of attraction $= \frac{B\ C}{b^2} = \frac{8 \times 4}{4} = 8.$

Between A and C, force of attraction $= \frac{A\ C}{c^2} = \frac{16 \times 4}{9} = 7.1.$

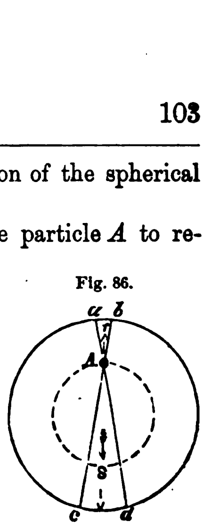
The attraction between the two bodies A and C is not interfered with by the body B; nor will other bodies stationed or moving between or about either one or both the masses A and C influence the force of attraction between A and C. Therefore, the force of attraction between any two particles of matter, whether embodied in a solid or porous mass or isolated in empty space, is equal to the product of the masses of the particles divided by the square of their distance apart.

§ 43. Suppose a particle of matter, A, Fig. 86, to be enclosed within a hollow material sphere, $a\ b\ c\ d$, the particle being a unit of

matter acted upon in all directions by the attraction of the spherical surface.

Draw the straight lines $a\ d$ and $b\ c$ through the particle A to represent the sides of the two cones with a common vertex at A. Let S represent the height of the large cone, and r that of the small one. $(a\ b) =$ diameter and $(a\ b)^2$ the area of the base of the small cone; $(c\ d) =$ diameter and $(c\ d)^2$ the area of the base of the large cone.

Fig. 86.



The particle A is attracted in opposite directions by the matter in the bases of the cones which constitute parts of the material spherical surface.

The attraction in the direction of the arrow r is $\frac{(a\ b)^2}{r^2}$.

In the direction of the arrow S the attraction is $\frac{(c\ d)^2}{S^2}$.

The angles of the sides of the two cones are alike, or the angles

$$a\ A\ b = c\ A\ d. \qquad \therefore\ (a\ b) : r = (c\ d) : S.$$

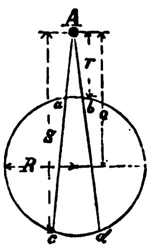
$$\frac{(a\ b)}{r} = \frac{(c\ d)}{S}, \qquad \frac{(a\ b)^2}{r^2} = \frac{(c\ d)^2}{S^2}.$$

That is to say, the opposing forces of attraction are alike or in equilibrium at the particle A, or that the particle is equally attracted from all sides by the spherical surface.

Now let the spherical shell be filled up with matter from the outer to the inner dotted concentric circle A; the equilibrium of attraction on the particle A will not be disturbed by that matter.

§ 44. If the particle A is located outside of the spherical shell, as represented by Fig. 87, the formulas of attractions will be the same as those for Fig. 86, but both the attractions will in this case act in one and the same direction on the particle, and the force of attraction will be
$$\frac{(a\ b)^2}{r^2} + \frac{(c\ d)^2}{S^2}.$$

Fig. 87.



Let e denote the distance of the particle from the centre of the sphere, and $R =$ radius of the sphere, then

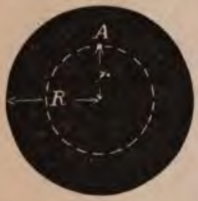
$$e = S - R, \qquad \text{but } S = e + R.$$
$$e = r + R, \qquad \text{and } r = e - R.$$

$$\frac{(a\ b)^2}{r^2} + \frac{(c\ d)^2}{S^2} = \frac{(a\ b)^2}{(e-R)^2} + \frac{(c\ d)^2}{e+R} = \frac{(a\ b)^2 + (c\ d)^2}{e^2}.$$

That is to say, if the matter of the bases of the cones $(a\,b)^2$ and $(c\,d)^2$ were located in the centre of the sphere, the force of attraction of that matter on the particle A would be the same as when at the surface of the sphere. Therefore, if the hollow sphere be filled up solid with matter, its force of attraction on the particle A would be equal to the mass of that matter divided by the square of the distance from the centre of the sphere to the particle.

§ 45. Fig. 88 represents a section of a solid sphere of homogeneous matter. A particle at A is attracted toward the centre only by the concentric sphere enclosed by the dotted circle. All attractions of the hollow sphere outside of the dotted circle are in equilibrium on the particle A, as proved by § 43.

Fig. 88.



Let R denote the radius of the whole sphere, and r = that of the inner sphere. The mass or quantity of matter in a sphere can be represented by R^3 or r^3. Then the force of attraction on the particle A will be $\frac{r^3}{r^2}=r$.

That is to say, the force of attraction on any particle of the matter in a solid sphere is proportionate to the distance r of that particle from the centre.

Suppose a hole to be made from the surface to the centre of the earth, and a body let down into it by a rope attached to a balance-scale; then the weight of the body will decrease with the depth until it reaches the centre of the earth, where it will indicate no weight on the balance-scale placed at the surface, omitting the weight of the rope.

r = radius of the earth.
d = any depth to which the body is sunk in the hole.
W = weight of the body at the surface of the earth, and
w = weight of the body at the depth d.

$$\text{Then } w=\frac{W(r-d)}{r}.$$

The radius r and depth d may be expressed in any unit of length, as well as W and w in any unit of weight.

The mean radius of the earth is about $r=3956$ miles, or $r=20887680$ feet.

When r and d are expressed in feet, the weight w will be

$$w=\frac{W(20887680-d)}{20887680}.$$

WEIGHT.

§ 46. The weight of a body is the force of attraction between the earth and that body. The weight of a body is greatest at the surface of the earth, and decreases above or below that surface. Above the surface the weight decreases as the square of its distance from the centre of the earth, and below the surface the weight decreases simply as its distance from the centre.

The weight of a body A, Fig. 87, weighing 100 pounds at the surface a, b of the earth, would weigh only 25 pounds at a height equal to the radius of the earth above a b.

A body A, Fig. 88, weighing 100 pounds at the surface of the earth, would weigh only 50 pounds at a depth of half the radius below the surface. Therefore, the weight of a body is not a constant quantity, whilst the quantity of matter in the body or the *mass* is constant wherever the body is weighed.

A wholly isolated body has no weight, but is an inert mass, incapable within itself of changing its own motion or rest. Any change in motion or rest of a body is derived from external force. The weight of a body is measured by the pressure it produces on its support. Two bodies in equilibrium on a balance-scale at the surface of the earth will also be in equilibrium above or below that surface, because the force of gravity acts equally on both bodies; but the force supporting the balance-scale varies in accordance with the law of gravity. A spring-balance will indicate the true weight of a body hung upon it wherever it is weighed.

The force of attraction between the earth and one gallon of distilled water at the level of the sea, in latitude of London, 51° 31′ N., is 10 pounds avoirdupois.

The standard English gallon contains 277.274 cubic inches. The temperature of the water and of the air in which it is weighed should be 62° Fahr., and the barometer 30 inches.

This is, however, not the true force of attraction between one gallon of water and the earth, for we must add the weight of 277.274 cubic inches of air which are displaced by the water. The weight of a cubic foot of dry air of temperature 62° Fahr. is 530 grains, which will be 0.01215 pounds for the capacity of one gallon. Therefore, when a gallon of water weighs 10 pounds in air the force of attraction between that water and the earth will be 10.01215 pounds.

MASS.

§ 47. **Mass** is the real quantity of matter in a body, and is proportioned to weight when compared in one or the same locality. Mass is a constant quantity, whilst weight varies with the force of gravity which produces it.

The force of gravity accelerates or increases the velocity of a falling body at the rate of 32 feet per second at the surface of the earth. This velocity is called the *acceleratrix of gravity*, and is generally denoted by the letter g.

When a force acts constantly on a body free to move, and the direction of the force passes through the centre of gravity of the body in the direction of motion, the velocity of the body will increase constantly as long as the force acts constantly.

Let M denote the mass of a body.

$F=$ the constant acting force.

$T=$ time of action.

$V=$ velocity of the body at the end of the time T.

These quantities bear the following relation to each other:

$$M : F = T : V, \quad \text{and} \quad M\,V = F\,T.$$

These functions are termed dynamic momentums, and distinguished as follows:

Momentum of motion $M\,V = F\,T$, momentum of time.

Neither of these momentums should be termed force.

When F is expressed in pounds, T in seconds and V in feet per second, then the unit of mass will be 32.17 pounds, which is equal in number to the acceleratrix g for a falling body at the surface of the earth.

MATT.

§ 48. No specific name has yet been given to any unit of mass, the want of which makes this subject somewhat obscure. Although we are told that mass is equal to the weight divided by the acceleratrix $g=32.17$, it does not make the same impression as if we had a specific name for the unit of mass, for which reason it is proposed to assign a name to it—namely, *matt.*, from the word matter; that is to say, one *matt.* $=32.17$ pounds, or the mass expressed in *matts.* multiplied by 32.17 would give the weight of the mass in pounds.

There are 69.63 matts. in a ton weight of 2240 pounds of matter.

If W denotes the weight of a body in pounds, then its mass expressed in *matts.* will be

$$M = \frac{W}{g} = \frac{W}{32.17}.$$

One matt. = the mass of 891 cubic inches of distilled water of temperature 40° Fahr.

The adoption of this term *matt.* will distinguish mass from force. Although the weight of a mass is force of gravity, all forces are not weights of matter. The force which sets in motion a railroad train is independent of the force of gravity, but may be as well expressed by weight, as the mass of the train; but we cannot solve the dynamical action without converting the weight or mass of the train into matts.

This is the fundamental principle of dynamics of matter, which should be distinctly understood and remembered, and is of so great importance that it is well worthy of repeating—namely,

$$M : F = T : V, \text{ and } MV = FT.$$

We have heretofore been taught, in text-books and in colleges, that momentum MV is force, which is a great error.

Force is only one element of momentum. Dynamic momentum divided by time is force; that is to say, if a mass $M = 4$ matts. moves with a velocity of $V = 6$ feet per second, its momentum is 24. If a force is applied to stop the mass, and can do so in $T = 3$ seconds, then the force of that momentum is $24 : 3 = 8$ pounds. Mass is inert, or incapable of changing its own motion or rest, and can, therefore, not be considered as force whether in motion or at rest.

Force is required in bringing a mass from rest to motion, or from motion to rest; but in either case that force must be applied from external causes independent of the mass.

When a body in motion is suddenly stopped, like that of a falling body striking the ground, it is stopped by the force of resistance it meets with, and not by any force within itself. The inertia of a body free to move presents a resistance equal to any force applied on it, whether in motion or at rest.

§ 49. TABLE FOR THE CONVERSION OF WEIGHT AND MASS.

The following table is for converting weight into mass, or mass into weight, which will be very useful in examples of dynamics of matter.

Example. Weight 957 pounds = 29.748 matts. See Table.

Example. Convert the mass of $M = 3466$ matts. into pounds.

	Matts.	= pounds.
	3460	= 111310
	6	= 193
Matts.	3466	= 111503 pounds.

	UNITS OF POUNDS.									
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Lbs.	matts.	matts.	matts.	matts.	matts.	matts.	matts.	matts.	matts.	matts.
0		.03108	.06217	.09325	.12434	.15542	.18651	.21759	.24868	.27976
10	.31085	.34193	.37302	.40410	.43519	.46627	.49736	.52844	.55953	.59061
20	.62170	.65278	.68387	.71495	.74604	.77712	.80821	.83929	.87038	.90146
30	.93254	.96363	.99472	1.0258	1.0569	1.0880	1.1191	1.1501	1.1812	1.2123
40	1.2434	1.2745	1.3056	1.3467	1.3678	1.3989	1.4300	1.4610	1.4921	1.5232
50	1.5542	1.5853	1.6164	1.6575	1.6786	1.7097	1.7408	1.7718	1.8029	1.8340
60	1.8651	1.8962	1.9273	1.9684	1.9895	2.0206	2.0517	2.0827	2.1138	2.1449
70	2.1759	2.2070	2.2381	2.2792	2.3003	2.3314	2.3625	2.3935	2.4246	2.4557
80	2.4868	2.5179	2.5490	2.5801	2.6112	2.6423	2.6734	2.7044	2.7355	2.7666
90	2.7976	2.8287	2.8598	2.8909	2.9220	2.9531	2.9842	3.0152	3.0463	3.0774
100	3.1085	3.1396	3.1707	3.2018	3.2329	3.2640	3.2951	3.3261	3.3572	3.3883
110	3.4193	3.4504	3.4815	3.5126	3.5437	3.5748	3.6059	3.6369	3.6680	3.6991
120	3.7202	3.7612	3.7924	3.8235	3.8546	3.8856	3.9168	3.9478	3.9789	4.0100
130	4.0310	4.0720	4.1032	4.1343	4.1654	4.1964	4.2276	4.2586	4.2897	4.3208
140	4.3419	4.3829	4.4141	4.4452	4.4763	4.5073	4.5385	4.5695	4.6006	4.6317
150	4.6527	4.6937	4.7249	4.7560	4.7871	4.8181	4.8493	4.8803	4.9114	4.9425
160	4.9636	5.0046	5.0358	5.0669	5.0980	5.1290	5.1602	5.1912	5.2223	5.2533
170	5.2744	5.3154	5.3466	5.3777	5.4088	5.4398	5.4710	5.5020	5.5331	5.5641
180	5.5853	5.6263	5.6575	5.6886	5.7197	5.7507	5.7819	5.8129	5.8440	5.8750
190	5.8961	5.9371	5.9683	5.9994	6.0305	6.0615	6.0927	6.1237	6.1548	6.1858
200	6.2170	6.2480	6.2792	6.3103	6.3414	6.3724	6.4036	6.4346	6.4657	6.4967
210	6.5278	6.5588	6.5900	6.6211	6.6522	6.6832	6.7144	6.7454	6.7765	6.8075
220	6.8387	6.8697	6.9009	6.9320	6.9631	6.9941	7.0253	7.0563	7.0874	7.1184
230	7.1495	7.1805	7.2117	7.2428	7.2739	7.3049	7.3361	7.3671	7.3982	7.4292
240	7.4604	7.4914	7.5226	7.5537	7.5848	7.6158	7.6470	7.6780	7.7091	7.7401
250	7.8612	7.8022	7.8334	7.8645	7.8956	7.9266	7.9578	7.9888	8.0199	8.0509
260	8.1721	8.1131	8.1443	8.1754	8.2065	8.2375	8.2687	8.2997	8.3308	8.3618
270	8.3930	8.4239	8.4552	8.4863	8.5174	8.5484	8.5796	8.6106	8.6417	8.6727
280	8.7038	8.7347	8.7660	8.7971	8.8282	8.8592	8.8904	8.9214	8.9525	8.9835
290	9.0146	9.0455	9.0768	9.1079	9.1390	9.1700	9.2012	9.2322	9.2633	9.2943
300	9.3255	9.3564	9.3877	9.4188	9.4499	9.4809	9.5121	9.5431	9.5742	9.6052
310	9.6363	9.6672	9.6985	9.7296	9.7607	9.7917	9.8229	9.8539	9.8850	9.9160
320	9.9472	9.9781	10.009	10.040	10.072	10.102	10.133	10.165	10.196	10.227
330	10.258	10.289	10.320	10.351	10.383	10.413	10.444	10.476	10.507	10.538
340	10.569	10.600	10.631	10.662	10.694	10.724	10.755	10.787	10.818	10.849
350	10.880	10.911	10.942	10.973	11.005	11.035	11.066	11.098	11.129	11.160
360	11.191	11.222	11.253	11.284	11.316	11.346	11.377	11.409	11.440	11.471
370	11.501	11.532	11.563	11.594	11.626	11.656	11.687	11.719	11.750	11.781
380	11.812	11.843	11.874	11.905	11.937	11.967	11.998	12.030	12.061	12.092
390	12.123	12.154	12.185	12.216	12.248	12.278	12.309	12.341	12.372	12.403
400	12.434	12.465	12.496	12.527	12.559	12.589	12.620	12.652	12.683	12.714
410	12.745	12.776	12.807	12.838	12.870	12.900	12.931	12.963	12.994	13.025
420	13.056	13.087	13.118	13.149	13.181	13.211	13.242	13.274	13.305	13.336
430	13.367	13.398	13.429	13.460	13.492	13.522	13.553	13.585	13.616	13.647
440	13.678	13.709	13.740	13.771	13.803	13.833	13.864	13.896	13.927	13.958
450	13.989	14.020	14.051	14.082	14.113	14.144	14.175	14.207	14.238	14.269
460	14.300	14.331	14.362	14.393	14.424	14.455	14.486	14.518	14.549	14.580
470	14.610	14.642	14.673	14.704	14.735	14.766	14.797	14.829	14.860	14.891
480	14.922	14.953	14.984	15.015	15.046	15.077	15.108	15.140	15.171	15.202
490	15.232	15.264	15.295	15.396	15.357	15.388	15.419	15.451	15.482	15.513
500	15.542	15.574	15.605	15.636	15.667	15.698	15.729	15.760	15.791	15.823

	UNITS OF POUNDS.									
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Lbs.	matts.	matts.	matts.	matts.	matts.	matts.	matts.	matts.	matts.	matts.
500	15.542	15.574	15.605	15.636	15.667	15.698	15.729	15.760	15.791	15.823
510	15.852	15.884	15.915	15.946	15.977	16.008	16.039	16.070	16.101	16.133
520	15.162	16.194	16.225	16.256	16.287	16.318	16.349	16.380	16.411	16.443
530	16.473	16.505	16.536	16.566	16.598	16.629	16.660	16.691	16.722	16.754
540	16.784	16.816	16.847	16.877	16.909	16.940	16.971	17.002	17.033	17.065
550	17.095	17.127	17.158	17.188	17.220	17.251	17.282	17.313	17.344	17.376
560	17.406	17.438	17.469	17.499	17.531	17.562	17.593	17.624	17.655	17.687
570	17.717	17.748	17.779	17.810	17.842	17.872	17.903	17.934	17.966	17.997
580	18.028	18.059	18.090	18.121	18.153	18.183	18.214	18.245	18.277	18.308
590	18.339	18.370	18.401	18.432	18.464	18.494	18.525	18.556	18.588	18.619
600	18.651	18.682	18.712	18.744	18.775	18.806	18.837	18.868	18.899	18.930
610	18.962	18.993	19.023	19.055	19.086	19.117	19.148	19.179	19.210	19.241
620	19.273	19.304	19.334	19.366	19.397	19.428	19.459	19.490	19.521	19.552
630	19.584	19.615	19.646	19.677	19.708	19.739	19.770	19.801	19.832	19.863
640	19.895	19.926	19.957	19.988	20.019	20.040	20.081	20.112	20.143	20.174
650	20.206	20.237	20.268	20.299	20.330	20.351	20.392	20.423	20.454	20.485
660	20.517	20.548	20.579	20.610	20.641	20.662	20.703	20.734	20.765	20.796
670	20.828	20.859	20.890	20.921	20.952	20.973	21.014	21.045	21.076	21.107
680	21.139	21.170	21.201	21.232	21.263	21.384	21.325	21.356	21.387	21.418
690	21.449	21.480	21.511	21.542	21.573	21.694	21.635	21.666	21.697	21.728
700	21.759	21.790	21.821	21.852	21.883	21.914	21.945	21.976	22.007	22.038
710	22.070	22.101	22.132	22.163	22.194	22.225	22.256	22.287	22.318	22.349
720	22.381	22.412	22.443	22.474	22.505	22.536	22.567	22.598	22.629	22.660
730	22.692	22.723	22.754	32.785	22.816	22.847	22.878	22.909	22.940	22.971
740	23.003	23.034	23.065	23.096	23.127	23.158	23.189	23.220	23.251	23.282
750	23.314	23.345	23.376	23.407	23.438	23.469	23.500	23.531	23.562	23.593
760	23.625	23.656	23.687	23.718	23.749	23.780	23.811	23.842	23.873	23.904
770	23.936	23.967	23.998	24.029	24.060	24.091	24.122	24.153	24.184	24.215
780	24.246	24.277	24.318	24.339	24.370	24.401	24.432	24.463	24.494	24.525
790	24.557	24.588	24.629	24.650	24.681	24.712	24.743	24.774	24.805	24.836
800	24.868	24.899	24.930	24.961	24.992	25.023	25.054	25.085	25.115	25.147
810	25.179	25.210	25.241	25.272	25.303	25.334	25.365	25.396	25.427	25.458
820	25.490	25.521	25.552	25.583	25.614	25.645	25.676	25.707	25.738	25.769
830	25.801	25.832	25.863	25.894	25.925	25.956	25.987	26.018	26.049	26.080
840	26.112	26.143	26.174	26.205	26.236	26.267	26.298	26.329	26.360	26.391
850	26.423	26.454	26.485	26.516	26.547	26.578	26.609	26.640	26.671	26.702
860	26.734	26.765	26.796	26.827	26.858	26.889	26.920	26.951	26.982	27.013
870	27.045	27.076	27.107	27.138	27.169	27.200	27.231	27.262	27.293	27.324
880	27.356	27.387	27.418	27.449	27.480	27.511	27.542	27.573	27.604	27.635
890	27.666	27.697	27.728	27.759	27.790	27.821	27.852	27.883	28.914	27.945
900	27.976	28.008	28.039	28.070	28.101	28.132	28.163	28.194	28.225	28.256
910	28.287	28.318	28.349	28.380	28.411	28.442	28.473	28.504	28.535	28.566
920	28.598	28.629	28.660	28.691	28.722	28.753	28.784	28.815	28.846	28.877
930	28.909	28.930	28.971	29.002	29.033	29.064	29.095	29.126	29.157	29.188
940	29.220	29.241	29.282	29.313	29.344	29.375	29.406	29.437	29.468	29.499
950	29.531	29.552	29.593	29.624	29.655	29.686	29.717	29.748	29.779	29.810
960	29.842	29.863	29.904	29.935	29.966	29.997	30.028	30.059	30.090	30.121
970	30.153	30.174	30.215	30.246	30.277	30 308	30.339	30.370	30.401	30.432
980	30.454	30.485	30.526	30.557	30.588	30.619	30.640	30.681	30.711	30.743
990	30.765	30.796	30.837	30.868	30.899	30.930	30.961	30.992	31.023	31.054
1000	31.085	31.107	31.148	31.179	31.210	31.241	31.272	31.303	31.334	31.365

	UNITS OF MATTS.									
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Matts	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
0		32.17	64.34	96.51	128.68	160.85	193.02	225.19	257.36	289.53
10	321.7	353.9	386.0	418.2	450.4	452.5	514.7	546.9	578.0	611.2
20	643.4	675.6	707.7	739.9	772.1	804.2	836.4	868.6	899.7	932.9
30	965.1	997.3	1029.4	1061.6	1093.8	1125.9	1158.1	1190.3	1221.4	1254.6
40	1286.8	1319.0	1351.1	1383.3	1415.5	1447.6	1479.8	1512.0	1543.1	1576.3
50	1608.5	1640.7	1672.8	1705.0	1737.2	1769.3	1801.5	1833.7	1864.8	1898.0
60	1930.2	1962.4	1994.5	2026.7	2058.9	2091.0	2123.2	2155.4	2186.5	2219.7
70	2251.9	2284.1	2316.2	2348.4	2380.6	2412.7	2444.9	2477.1	2508.2	2541.4
80	2573.6	2605.9	2637.9	2670.1	2702.3	2734.4	2766.6	2798.9	2830.0	2863.1
90	2895.3	2927.6	2959.6	2991.8	3024.0	3056.1	3088.3	3120.6	3152.6	3184.8
100	3217	3249.3	3281.3	3313.5	3345.7	3377.8	3410.0	3442.3	3474.3	3506.5
110	3538.7	3571.0	3603.0	3635.2	3667.4	3699.5	3731.7	3764.0	3796.0	3828.2
120	3860.4	3892.7	3924.7	3956.9	3989.1	4021.2	4053.4	4085.7	4117.7	4149.9
130	4182.1	4214.4	4246.4	4278.6	4310.8	4342.9	4375.1	4407.4	4439.4	4471.6
140	4503.8	4536.0	4568.1	4600.3	4632.5	4664.6	4696.8	4729.1	4761.1	4793.3
150	4825.5	4857.7	4889.8	4922.0	4954.2	4986.3	5018.5	5050.8	5082.8	5115.0
160	5147.2	5179.4	5211.5	5243.7	5275.9	5308.0	5340.2	5372.5	5404.5	5436.7
170	5468.9	5501.1	5533.2	5565.4	5597.6	5629.7	5661.9	5694.2	5726.2	5758.4
180	5790.6	5822.8	5854.9	5887.1	5919.3	5951.4	5983.6	6015.9	6047.9	6080.1
190	6112.3	6144.5	6176.6	6208.8	6241.0	6273.1	6305.3	6337.6	6369.6	6401.8
200	6434.0	6466.2	6498.3	6530.5	6562.7	6594.8	6627.0	6659.3	6691.3	6723.5
210	6755.7	6787.9	6820.0	6852.2	6884.4	6916.5	6948.7	6981.0	7013.0	7045.2
220	7077.4	7109.6	7141.7	7173.9	7206.1	7238.2	7270.4	7302.7	7334.7	7366.9
230	7399.1	7531.3	7463.4	7495.6	7527.8	7559.9	7492.1	7624.4	7656.4	7688.6
240	7720.8	7753.0	7785.1	7817.3	7849.5	7881.6	7913.8	7946.1	7978.1	8010.3
250	8042.5	8074.7	8106.8	8139.0	8171.2	8203.3	8235.5	8267.8	8299.8	8332.0
260	8364.2	8396.4	8428.5	8460.7	8492.9	8525.0	8557.2	8589.5	8621.5	8653.7
270	8685.9	8718.1	8750.2	8782.4	8814.6	8846.7	8878.9	8911.1	8943.2	8975.4
280	9007.6	9039.8	9071.9	9104.1	9136.3	9168.4	9200.6	9232.8	9264.9	9297.1
290	9329.3	9361.5	9393.6	9425.8	9458.0	9490.1	9522.3	9554.5	9586.6	9618.8
300	9651.0	9683.2	9715.3	9747.5	9779.7	9811.8	9844.0	9876.2	9908.3	9940.5
310	9972.7	10005	10037	10069	10101	10133	10166	10198	10230	10262
320	10294	10327	10359	10381	10423	10455	10488	10520	10552	10584
330	10616	10649	10681	10703	10745	10777	10810	10842	10874	10906
340	10938	10970	11002	11034	11066	11098	11131	11163	11195	11227
350	11259	11292	11324	11356	11388	11420	11453	11485	11517	11549
360	11581	11614	11646	11678	11710	11742	11775	11807	11839	11871
370	11903	11936	11968	12000	12032	12064	12097	12129	12161	12193
380	12225	12257	12289	12321	12353	12385	12418	12450	12482	12514
390	12547	12579	15611	12643	12675	12707	12740	12772	12804	12836
400	12868	12901	12933	12965	12997	13029	13062	13094	13126	13158
410	13189	13222	13254	13286	13318	13350	13383	13415	13447	13479
420	13510	13543	13575	13607	13639	13671	13704	13736	13768	13800
430	13832	13865	13897	13929	13961	13993	14026	14058	14090	14122
440	14154	14187	14219	14251	14283	15315	14348	14380	11412	14444
450	14476	14508	14540	15572	14604	14636	14669	14701	14733	14765
460	14798	14830	14862	14894	14926	14958	14991	15023	15055	15087
470	15120	15152	15184	15216	15248	15280	15313	15345	15377	15409
480	15441	15473	15505	15537	15570	15602	15634	15666	15698	15731
490	15763	15795	15827	15859	15892	15924	15956	15988	16020	16053
500	16085	16117	16149	16181	16214	16246	16278	16310	16342	16375

	UNITS OF MATTS.									
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Matts	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
500	16085	16117	16149	16181	16214	16246	16278	16310	16342	16375
510	16406	16438	16470	16502	16535	16567	16599	16631	16663	16696
520	16728	16760	16792	16824	16857	16889	16921	16953	16985	17018
530	17050	17082	17114	17146	17179	17211	17243	17275	17307	17340
540	17372	17404	17436	17468	17501	17533	17565	17597	17629	17662
550	17693	17725	17757	17789	17822	17854	17886	17918	17950	17983
560	18015	18047	18079	18111	18144	18176	18208	18240	18272	18305
570	18337	18369	18401	18433	18466	18498	18530	18562	18594	18627
580	18659	18691	18723	18755	18798	18820	18852	18884	18916	18949
590	18980	19012	19044	19076	19109	19141	19173	19205	19237	19270
600	19302	19334	19366	19398	19431	19463	19495	19527	19559	19592
610	19624	19656	19688	19720	19753	19785	19817	19849	19881	19914
620	19946	19978	20010	20042	20075	20107	20139	20171	20203	20236
630	20267	20299	20331	20363	20396	20428	20460	20492	20524	20557
640	20589	20621	20653	20685	20718	20750	20782	20814	20846	20879
650	20911	20943	20975	21007	21040	21072	21104	21136	21168	21201
660	21233	21265	21297	21329	21362	21394	21426	21458	21490	21523
670	21554	21586	21618	21650	21683	21715	21747	21779	21811	21844
680	21876	21908	21940	21972	22005	22037	22069	22101	22133	22166
690	22198	22230	22262	22294	22327	22359	22391	22423	22455	22488
700	22519	22551	22583	22615	22648	22680	22712	22744	22776	22809
710	22841	22873	22905	22937	22970	23002	23034	23066	23098	23131
720	23163	23194	23227	23259	23292	23324	23356	23388	23419	23453
730	23484	23516	23548	23580	23613	23645	23677	23709	23741	23774
740	23806	23838	23870	23902	23935	23967	23999	24031	24064	24096
750	24128	24160	24192	24224	24257	24289	24321	24353	24386	24418
760	24450	24482	24514	24546	24579	24611	24643	24675	24708	24740
770	24771	24803	24835	24867	24900	24932	24964	24996	25029	25061
780	25093	25125	25157	25189	25222	25254	25286	25318	25350	25383
790	25415	25447	25479	25511	25544	25576	25608	25640	25671	25705
800	25736	25768	25800	25833	25865	25897	25929	25961	25993	26026
810	26057	26089	26121	26154	26186	26218	26250	26282	26314	26347
820	26379	26411	26443	26476	26508	26540	26572	26604	26636	26669
830	26701	26733	26765	26798	26830	26862	26894	26926	26958	26991
840	27023	27055	27087	27120	27152	27184	27216	27248	27280	27313
850	27344	27376	27408	27441	27473	27505	27537	27569	27601	27634
860	27666	27698	27730	27763	27795	27827	27859	27891	27923	27956
870	27988	28020	28052	28085	28117	28149	28181	28213	28245	28278
880	28309	28341	28373	28406	28438	28470	28502	28534	28566	28599
890	28631	28663	28695	28728	28760	28792	28824	28856	28888	28921
900	28953	28985	29017	29050	29082	29114	29146	29178	29210	29243
910	29275	29307	29339	29372	29404	29436	29468	29500	29532	29565
920	29596	29628	29660	29693	29725	29757	29789	29821	29853	29886
930	29918	29950	29982	30015	30047	30079	30111	30143	30175	30208
940	30240	30272	30304	30337	30369	30401	30433	30465	30497	30530
950	30561	30593	30625	30658	30690	30722	30754	30786	30818	30851
960	30883	30915	30947	30980	31012	31043	31076	31108	31140	31173
970	31205	31237	31269	31302	31334	31365	31398	31430	31462	31495
980	31527	31559	31591	31624	31656	31687	31720	31752	31784	31817
990	31848	31880	31912	31945	31977	32009	32041	32073	32105	32138
1000	32170	32202	32234	32267	32299	32331	32363	32395	32427	32460

§ 50. THE EARTH'S ATTRACTION ON ITS SURFACE.

The attraction or weight of bodies on the earth's surface is slightly influenced by various causes, namely: 1st. The flatness of the poles makes the radius shortest in the direction of the earth's axis, which increases the force of attraction at the poles.

The radius R of the earth in feet at any latitude L is about

$$R^2 = 20887680(1+0.00164 \cos. 2L).$$

2d. The centrifugal force on the surface of the earth, which varies as the cosine for the latitude and acts in opposition to the force of attraction, is as follows:

$$F = \frac{A\,R n^2 \cos.L^2}{2933.5},$$

in which A = unit of attraction at the radius R. n = revolutions per minute.

$$n^2 = \frac{1}{(60 \times 24)^2} = \frac{1}{2073600}.$$

3d. The centrifugal force of the earth's rotation around the sun influences the attraction as the *sine* of the sun's angle with the surface of the earth. This influence is so insignificant that it could not be detected in the most delicate scientific experiment yet known.

4th. The attraction of the sun and moon influences the attraction of bodies on the earth's surface directly as the *sine* of the sun's or moon's angle with the surface, and is demonstrated by the tidal wave, causing high and low water.

§ 51. MASS OF THE EARTH.

It is not necessary for the purpose of this treatise to observe the precision required in highly scientific investigations, for which reason we will limit ourselves within the supposition that the earth is a perfect sphere of radius $R = 20,887,680$ feet, and is at rest.

The delicate pendulum experiments of Cavendish approximated the mean density of the earth to be 5.44 times that of water. The weight of a cubic foot of distilled water of temperature 39° Fahr. is 62.388 pounds. Then the mass of the earth, expressed in matts., will be

$$M = \frac{4\pi R^3 \, 62.388 \times 5.44}{3 \times 32.17} = 402,735,000,000,000,000,000,000 \text{ matts.}$$

$$Log.\ M = 23.6050086.$$

Let r denote the radius of the earth expressed by such a unit of length that the force F of attraction will be

$$F = \frac{M\,m}{r^2} = 32.17,$$

which is the force of attraction between the earth and one matt. of matter on the surface of the earth.

Then we have that radius of the earth to be

$$r^2 = \frac{M\,m}{32.17}, \qquad \text{and } r = \sqrt{\frac{M\,m}{32.17}} = 111{,}886{,}700{,}000 \text{ units.}$$

$$Log.\ r = 11.0487787.$$

$$\frac{r}{R} = 5356.59, \qquad \text{and } \frac{r^2}{R^2} = 28{,}693{,}080.$$

$$Log. = 3.7288886. \qquad Log. = 7.4577772.$$

The force of attraction F in pounds between any two masses M and m, expressed in matts., and their distance apart D in feet, will be

$$F = \frac{M\,m}{28693080\ D^2}, \qquad \text{and } D = 5356.59\sqrt{\frac{M\,m}{F}}.$$

Example. Two bodies $M = 450$ and $m = 360$ matts. are held at a distance $D = 2$ feet apart. Required the force of attraction between them?

$$F = \frac{450 \times 360}{28693080 \times 2^2} = 0.0014115 \text{ of a pound.}$$

When the masses of the attracting bodies are expressed by weights W and w in pounds, we have

$$W = g\ M, \qquad \text{and } w = g\ m,$$

$$\text{or } M = \frac{W}{g}, \qquad \text{and } m = \frac{w}{g}.$$

$$28693080\ g^2 = 29{,}694{,}700{,}000, \qquad log. = 10.4726794.$$

$$\text{Force of attraction, } F = \frac{W\,w}{29{,}694{,}700{,}000\ D^2}.$$

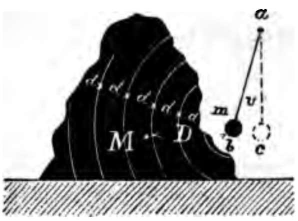
§ 52. ATTRACTION OF A MOUNTAIN.

The accompanying illustration represents a mountain of known quantity of matter M, at the side of which is hung on a string $a\ b$ a mass m. The dotted line $a\ c$ represents the vertical in which the body m would hang if not attracted by the mountain, and $a\ b$ is the resultant of the two forces of attraction from the earth and mountain.

Let v denote the angle b, a, c formed by the pendulum, then the *cosine* for v represents the earth's attraction, and *sin.v* that of the mountain.

W = weight of the body m, which is the force of attraction in the direction a, c, and F = force of attraction of the mountain, we have

Fig. 89.



$$F : W = sin.v : cos.v.$$

$$\frac{F}{W} = \frac{sin.\ v}{cos.\ v} = tan.v.$$

$$F = W\ tan.v, \quad \text{and} \quad W = F\ cot.v.$$

$$tan.v = \frac{F}{W}.$$

To find the distance D between the body m and the centre of attraction of the mountain is a rather complicated problem, which, however, can be approximated by knowing the form of the mountain.

With the body m as a centre draw concentric circles equal distances d apart through the mountain, representing spherical sections of areas e, f, g, h, etc., and call A = the sum of all the spherical sections.

Reduce each area to a zone of the same radius, and multiply it with *cosine* for ½ the angle occupied by that zone; the product will be the areas e, f, g, h, to be inserted in the following formulas:

Then, $$\frac{A}{D^2} = \frac{e}{d^2} + \frac{f}{(2\,d)^2} + \frac{g}{(3\,d)^2} + \frac{h}{(4\,d)^2},$$

and $$D = d\sqrt{\frac{A}{\frac{e}{1} + \frac{f}{4} + \frac{g}{9} + \frac{h}{16} + \text{, etc.}}}.$$

The base-line of the mountain need not be referred to the level of the sea, but to the level of the ground surrounding it.

Example 1. The quantity of matter in a mountain is estimated to

be $M = 1{,}250{,}000{,}000{,}000$ matts.; the distance between the mass m and centre of attraction of the mountain is approximated to $D = 5000$ feet; and the weight of the body $W = 321.7$ or the mass $m = 10$ matts. Required the force of attraction between the body m and mountain M?

$$\text{Formula, } F = \frac{1{,}250{,}000{,}000{,}000 \times 10}{28693080 \times 5000^2} = 0.0175 \text{ pounds.}$$

Example 2. In connection with the preceding example the body m is hung on a string 20 feet long. Required the angle v and the linear deviation a, b of the body?

$$\text{Formula, } tan.v = \frac{F}{W} = \frac{0.0175}{321.7} = 0.0000542,$$

which corresponds to an angle of $v = 0° \, 0' \, 11''$. The linear deviation a, b will be $a, b = 20 \times sin.v = 0.0011$ feet $= 0.0132$ inches.

An artificial horizon at the place of the body m would have the same inclination.

§ 53. INERTIA.

Inertia is the force of resistance which a body presents to any external force tending to change the motion or rest of that body.

An isolated rigid body is incapable within itself of changing its own state of motion or rest, which condition is called *inert.*

A system of bodies by their own virtues of attraction may be capable of changing each other's motion or rest, as is the case with the heavenly bodies, which by their combined attractions regulate one another's motion in space.

A falling body does not move simply by the earth's attraction alone, but by the combined virtues of attraction between the two bodies.

If a body had no virtue of attraction, no other body could exercise any attraction on it, but there exists no ponderable matter without the virtue of universal attraction.

If a body on the earth's surface could be deprived of its virtue of universal attraction, the earth's attraction would have no effect upon it, and that body would have no weight, but could be placed at rest anywhere in space without support.

Two bodies of iron attract one another by their virtue of magnetic attraction, but a body of iron exercises no magnetic attraction on a body of copper, because the latter possesses no such virtue of attraction.

§ 54. FALLING BODIES.

Falling bodies are drawn towards the earth by the force of gravity or the earth's attraction, which force is equal to the weight of the body.

$$F : M = V : T,$$

But $F = W$, and $M = \frac{W}{g}$,

therefore $W : \frac{W}{g} = V : T$ of which $V = g\,T$.

That is to say, the velocity of a falling body in feet per second at the end of fall is equal to the time of fall multiplied by the acceleratrix $g = 32.17$, which shows that the body gains a velocity of 32.17 feet per second for every second of its time of continued fall.

The space fallen through will be equal to the time of fall, multiplied by the mean velocity $\frac{V}{2}$.

Let S denote the space in feet, we have

$$S = \frac{V\,T}{2} = \frac{g\,T^2}{2} = \frac{V^2}{2g} = \frac{V^2}{64.34}.$$

By solving these formulas we have the time of fall,

$$T = \frac{V}{g} = \frac{2S}{V} = \sqrt{\frac{2S}{g}} = \frac{VS}{4.01}.$$

Let u denote the space fallen through in the Tth second, we have

$$u = g\,(T - \tfrac{1}{2}) \text{ and } T = \frac{u}{g} + \tfrac{1}{2}.$$

§ 55. FALL OF HEAVY AND LIGHT BODIES.

In vacuum a heavy body does not fall faster than a light one, because the weight of each body is equal to the force of gravity acting upon it; but when a body falls in air or in a liquid, its force of gravity is diminished equal to the weight of the air or liquid displaced by the body; and whilst the mass is constant, a smaller force has a heavier body to move, and the body will fall slower. A pound of lead dis-

places less weight of air than does a pound of cork, for which reason the lead will fall faster than the cork in air.

The force of gravity must also overcome the resistance of the air to the motion of the falling body, which is independent of the weight of air the body displaces. This resistance increases as the square of the velocity and as the surface exposed to the motion. A pound of cork exposes more surface to the motion than does a pound of lead, for which reason the cork falls much slower.

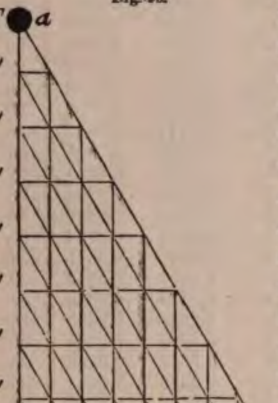
Formulas with these influences on falling bodies will be given farther on.

§ 56. ILLUSTRATION FOR FALLING BODIES.

The illustration represents the velocity V, space S and time T of a body falling from a to b, with the corresponding numbers under the respective heads. In the first second, the body falls from T to 1″, a space of 16.085 feet, which is represented by the first triangle, and which is the unit of the system. The velocity attained in this period is $V = 2 \times 16.085 = 32.17$ feet per second. If the force of gravity ceased to act at the end of the first second, the body would continue to fall with a uniform velocity of 32.17 feet per second; but as the force of gravity acts constantly, the body will attain an additional velocity of 32.17 feet for every second of its fall.

Fig. 90.

u	V	S	T
1	2	1	1″
3	4	4	2″
5	6	9	3″
7	8	16	4″
9	10	25	5″
9	12	36	6″
13	14	49	7″
15	16	64	8″



Between 1″ and 2″ the body passes 3 triangles, which is the space $3 \times 16.085 = 48.255$ feet passed through in the second second; and the velocity at 2″ is $4 \times 16.085 = 64.34$ feet per second.

In the eighth second of its fall it passes 15 triangles, which correspond to $15 \times 16.085 = 241.275$ feet between 7″ and 8″; and the velocity at b is $16 \times 16.085 = 257.36$ feet per second.

From a to b the body passed 64 triangles, which represents the linear space from a to b, or $64 \times 16.085 = 1029.44$ feet.

FALLING BODIES.

V = velocity in feet per second at the end of fall.
T = time in seconds of the fall.
S = space fallen through in feet.

V	T	S	V	T	S	V	T	S
0.1	0.0031	.00015	5.1	0.1585	0.4042	11	0.3419	1.8804
0.2	0.0062	.00031	5.2	0.1616	0.4202	12	0.3730	2.2380
0.3	0.0093	0.0014	5.3	0.1647	0.4364	13	0.4041	2.6266
0.4	0.0124	0.0025	5.4	0.1678	0.4530	14	0.4352	3.0464
0.5	0.0155	0.0039	5.5	0.1709	0.4700	15	0.4663	3.4975
0.6	0.0186	0.0055	5.6	0.1740	0.4872	16	0.4973	3.9784
0.7	0.0217	0.0076	5.7	0.1771	0.5047	17	0.5284	4.4914
0.8	0.0248	0.0099	5.8	0.1802	0.5226	18	0.5595	5.0355
0.9	0.0279	0.0125	5.9	0.1833	0.5407	19	0.5906	5.6107
1.	0.0311	0.0155	6.	0.1865	0.5595	20	0.6217	6.2170
1.1	0.0342	0.0188	6.1	0.1896	0.5782	21	0.6527	6.8502
1.2	0.0373	0.0224	6.2	0.1927	0.5973	22	0.6838	7.5218
1.3	0.0404	0.0262	6.3	0.1958	0.6168	23	0.7149	8.2213
1.4	0.0435	0.0304	6.4	0.1989	0.6365	24	0.7460	8.9520
1.5	0.0446	0.0335	6.5	0.2020	0.6565	25	0.7771	9.7125
1.6	0.0477	0.0381	6.6	0.2051	0.6768	26	0.8082	10.566
1.7	0.0508	0.0432	6.7	0.2082	0.6975	27	0.8393	11.330
1.8	0.0539	0.0485	6.8	0.2113	0.7184	28	0.8704	12.185
1.9	0.0580	0.0551	6.9	0.2144	0.7397	29	0.9015	13.072
2.	0.0622	0.0622	7.	0.2176	0.7616	30	0.9325	13.987
2.1	0.0653	0.0685	7.1	0.2207	0.7835	31	0.9636	14.936
2.2	0.0684	0.0756	7.2	0.2238	0.8057	32	0.9947	15.915
2.3	0.0715	0.0822	7.3	0.2269	0.8282	33	1.0258	16.926
2.4	0.0746	0.0895	7.4	0.2300	0.8510	34	1.0569	17.967
2.5	0.0777	0.0971	7.5	0.2331	0.8741	35	1.0879	19.038
2.6	0.0808	0.1050	7.6	0.2362	0.8975	36	1.1190	20.142
2.7	0.0839	0.1135	7.7	0.2393	0.9213	37	1.1501	21.277
2.8	0.0870	0.1218	7.8	0.2424	0.9453	38	1.1812	22.443
2.9	0.0901	0.1305	7.9	0.2455	0.9697	39	1.2123	23.640
3.	0.0932	0.1398	8.	0.2487	0.9948	40	1.2434	24.868
3.1	0.0963	0.1492	8.1	0.2518	1.0168	41	1.2745	26.127
3.2	0.0994	0.1590	8.2	0.2549	1.0451	42	1.3056	27.417
3.3	0.1025	0.1691	8.3	0.2580	1.0707	43	1.3367	28.739
3.4	0.1054	0.1795	8.4	0.2611	1.0966	44	1.3678	29.407
3.5	0.1087	0.1886	8.5	0.2642	1.1228	45	1.3989	31.475
3.6	0.1118	0.2012	8.6	0.2673	1.1494	46	1.4300	32.890
3.7	0.1149	0.2125	8.7	0.2704	1.1762	47	1.4611	34.336
3.8	0.1170	0.2223	8.8	0.2735	1.2034	48	1.4922	35.813
3.9	0.1201	0.2355	8.9	0.2766	1.2259	49	1.5233	37.321
4.	0.1243	0.2486	9.	0.2797	1.2586	50	1.5544	38.830
4.1	0.1274	0.2611	9.1	0.2828	1.2867	51	1.5854	40.413
4.2	0.1305	0.2740	9.2	0.2859	1.3151	52	1.6165	42.029
4.3	0.1336	0.2872	9.3	0.2890	1.3438	53	1.6475	43.659
4.4	0.1367	0.2939	9.4	0.2921	1.3729	54	1.6786	45.322
4.5	0.1398	0.3145	9.5	0.2952	1.4022	55	1.7097	47.017
4.6	0.1429	0.3286	9.6	0.2983	1.4318	56	1.7407	48.740
4.7	0.1460	0.3431	9.7	0.3014	1.4618	57	1.7718	50.396
4.8	0.1491	0.3578	9.8	0.3045	1.4920	58	1.8029	52.284
4.9	0.1522	0.3729	9.9	0.3076	1.5226	59	1.8340	54.103
5.	0.1554	0.3885	10.	0.3108	1.5540	60	1.8651	55.953

FALLING BODIES.

V = velocity in feet per second at the end of fall.
T = time in seconds of the fall.
S = space fallen through in feet.

V	T	S	V	T	S	V	T	S
65	2.0206	65.669	530	16.478	4366.6	1030	32.027	16494
70	2.1769	76.260	540	16.788	4452.8	1040	32.338	16815
75	2.3314	87.427	550	17.099	4701.7	1050	32.649	17141
80	2.4868	97.472	560	17.409	4874.5	1060	32.950	17463
85	2.6422	112.29	570	17.720	5050.2	1070	33.261	17794
90	2.7976	125.89	580	18.030	5228.7	1080	33.572	18129
95	2.9530	140.27	590	18.341	5410.6	1090	33.883	18446
100	3.1085	155.42	600	18.651	5595.3	1100	34.194	18806
110	3.4194	188.07	610	18.961	5783.1	1110	34.504	19149
120	3.7302	223.81	620	19.271	5974.0	1120	34.815	19496
130	4.0411	262.67	630	19.582	6168.3	1130	35.126	19846
140	4.3519	304.63	640	19.893	6365.7	1140	35.436	20198
150	4.6627	349.70	650	20.204	6566.3	1150	35.747	20504
160	4.9736	397.88	660	20.515	6770.0	1160	36.058	20913
170	5.2844	449.18	670	20.826	6976.7	1170	36.369	21275
180	5.5953	503.36	680	21.137	7186.6	1180	36.680	21641
190	5.9061	561.08	690	21.448	7399.5	1190	36.991	22009
200	6.2170	621.70	700	21.759	7615.6	1200	37.302	22381
210	6.5279	689.43	710	22.070	7834.8	1210	37.613	22755
220	6.8387	752.26	720	22.380	8056.8	1220	37.924	23133
230	7.1496	822.20	730	22.691	8282.2	1230	38.235	23514
240	7.4604	895.25	740	23.002	8510.7	1240	38.546	23898
250	7.7713	971.41	750	23.313	8742.4	1250	38.857	24285
260	8.0821	1050.6	760	23.623	8976.7	1260	39.168	24676
270	8.3930	1133.1	770	23.934	9214.6	1270	39.479	25069
280	8.7038	1218.5	780	24.245	9455.5	1280	39.780	25459
290	9.0147	1308.2	790	24.556	9699.6	1290	40.090	25855
300	9.3255	1398.8	800	24.868	9947.2	1300	40.411	26267
310	9.6363	1493.7	810	25.179	10197	1310	40.722	26673
320	9.9472	1591.6	820	25.490	10451	1320	41.033	27081
330	10.258	1690.6	830	25.801	10707	1330	41.343	27493
340	10.569	1791.7	840	26.112	10967	1340	41.654	27908
350	10.879	1903.8	850	26.423	11230	1350	41.965	28326
360	11.190	2014.2	860	26.733	11495	1360	42.276	28747
370	11.501	2127.7	870	27.044	11764	1370	42.587	29172
380	11.812	2244.3	880	27.354	12035	1380	42.897	29599
390	12.123	2364.0	890	27.665	12311	1390	43.208	30029
400	12.434	2486.8	900	27.976	12589	1400	43.519	30463
410	12.745	2612.7	910	28.287	12871	1410	43.820	30893
420	13.055	2741.5	920	28.598	13155	1420	44.131	31333
430	13.366	2873.7	930	28.908	13442	1430	44.442	31776
440	13.677	3008.9	940	29.219	13733	1440	44.753	32222
450	13.989	3144.8	950	29.530	14027	1450	45.064	32671
460	14.300	3289.0	960	29.841	14323	1460	45.375	33123
470	14.611	3433.6	970	30.152	14623	1470	45.686	33579
480	14.922	3581.3	980	30.463	14927	1480	45.997	34037
490	15.233	3732.1	990	30.774	15233	1490	46.308	34499
500	15.545	3886.2	1000	31.085	15542	1500	46.631	34973
510	15.856	4043.3	1010	31.396	15855	1510	46.732	35082
520	16.167	4203.4	1020	31.707	16179	1520	47.043	35752

§ 57. FORMULAS FOR BODIES FALLING FREELY UNDER THE ACTION OF GRAVITY.

Velocity in feet per second at the end of fall.	Space in feet fallen through in the time T.	Time of fall in seconds.	Space in feet fallen through in the Tth second.
$V.$	$S.$	$T.$	$u.$
$V=gT.$. 1	$S=\frac{gT^2}{2}.$. 5	$T=\frac{V}{g}.$. 9	$u=g(T-\frac{1}{2}).$ 13
$V=\frac{2S}{T}.$. 2	$S=\frac{VT}{2}.$. 6	$T=\frac{2S}{V}.$. 10	$T=\frac{u}{g}+\frac{1}{2}.$. 14
$V=\sqrt{2gS}.$. 3	$S=\frac{V^2}{2g}.$. 7	$T=\sqrt{\frac{2S}{g}}.$. 11	
$V=8.02\sqrt{S}.$. 4	$S=\frac{V^2}{64.34}.$. 8	$T=\frac{\sqrt{S}}{4.01}.$. 12	

Example. A body is dropped from a height of $S=80$ feet. Required its time of fall, and with what velocity it reaches the ground?

Time, Formula 11. $T=\sqrt{\frac{2\times80}{32.17}}=5$ seconds.

Velocity, Formula 4. $V=8.02\sqrt{80}=71.73$ feet per second.

§ 58. ASCENDING BODIES.

Ascending bodies are raised by external force superior to that of gravity. The start of a body from rest to a uniform ascending motion requires a greater force than that of gravity or weight of the body. The actual force which starts the body vertically upward is the difference between the applied force and the weight of the body.

The force required to maintain a uniform ascending motion is equal to the weight of the body. If the force suddenly ceases to act, the body will continue to ascend against the force of gravity with a retarded motion until it stops, when the force of gravity returns the body, which is then said to fall.

The retarded motion of a body ascending against the force of gravity is inversely the same as the accelerated motion in its descent.

Suppose the case of firing a ball vertically upward from a gun:

V = the velocity given to the ball at the muzzle.

v = the velocity at the time t from the time the ball left the muzzle of the gun.

T = time the ball will ascend.

t = any time less than T.

S = height in feet to which the ball will ascend.

s = the height the ball ascends in the time t.

Fig. 91.

In the accompanying Fig. 91 a represents the muzzle of a gun, at which the charge of powder has given an ascending velocity V to the ball. The height to which the ball will reach at c is found by Formula 7,

$$S=\frac{V^2}{2g}.$$

The time T required for the ascent will be Formula 9,

$$T=\frac{V}{g}.$$

On reaching c the force of gravity will return the ball to the muzzle a in an equal time T and with the same velocity V, so that the ascending retarded motion is inversely the same as the accelerated descending motion, or at equal heights above the muzzle, the ball will have equal velocities of ascension or descension.

If t denotes the time in which the body ascends from a to b, or descends from b to a, and T the time from a to c, or from c to a, then the time between a and c will be $T-t$, and the space or vertical height between b and c is $S-s$. Insert these values in Formula 11, which will be

$$T-t=\sqrt{\frac{2(S-s)}{g}}.$$

Formula 9, $$T=\frac{V}{g}, \quad \text{and} \quad S=\frac{V^2}{2g}.$$

Then we have $$\frac{V}{g}-t=\sqrt{\frac{2\left(\frac{V^2}{2g}-s\right)}{g}}.$$

$$t=\frac{V}{g}-\sqrt{\frac{V^2}{g}-\frac{2s}{g}}. \quad . \quad . \quad . \quad . \quad 22$$

This formula gives the time of ascension of the ball from a to b, or descension from b to a when falling from c.

The velocity v of the ball at b will be

Formula 3, $$v=\sqrt{2\,g\,(S-s)}.$$

$$S=\frac{V^2}{2\,g} \quad \text{and} \quad v=\sqrt{2\,g\left(\frac{V^2}{2\,g}-s\right)},$$

from which $$v=\sqrt{V^2-2\,g\,s}. \quad . \quad . \quad . \qquad 15$$

§ 59. FORMULAS FOR BODIES ASCENDING UNDER FREE ACTION OF GRAVITY.

Velocity v in feet per second at the time t or at the height s.

$$v=\sqrt{V^2-2\,g\,s}. \quad . \qquad 15$$

$$v=\frac{S}{t}-\frac{g\,t}{2}. \quad . \quad . \qquad 16$$

Height s in feet at the time t or when the velocity is v.

$$s=V\,t-\frac{g\,t^2}{2} \quad . \quad . \qquad 17$$

$$s=t\,v+\frac{g\,t^2}{2}. \quad . \quad . \qquad 18$$

The initial velocity V in feet per second with which the body starts to ascend under action of gravity.

$$V=v+g\,t. \quad . \quad . \quad . \qquad 19$$

$$V=\frac{s}{t}+\frac{g\,t}{2}. \quad . \quad . \quad . \qquad 20$$

Time t in seconds required to reach the height s, or the velocity reduced to v.

$$t=\frac{V-v}{g}. \quad . \quad . \quad . \qquad 21$$

$$t=\frac{V}{g}-\sqrt{\frac{V^2}{g^2}-\frac{2\,s}{g}}. \quad . \qquad 22$$

The formulas for T and S are the same as those for falling bodies.

Example. A body is forced vertically upward with a velocity of $V=560$ feet per second. What will be its velocity at a height of $s=420$ feet? and to what height will it ascend?

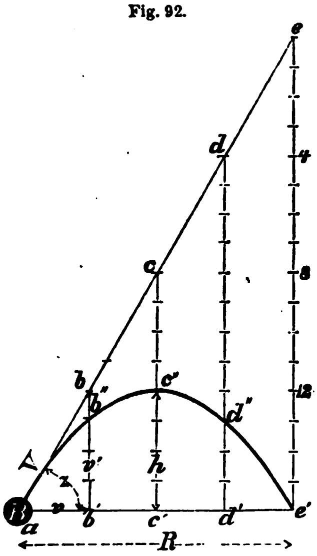
Formula 15, $v=\sqrt{560^2-2\times 32.17\times 420}=535.25$ feet per second, the velocity required.

Formula 7, $S=\dfrac{560^2}{2\times 32.17}=4874.1$ feet, the height required.

§ 60. PARABOLIC MOTION OF BODIES.

A body B thrown up from a in the inclined direction a, b, c, d, e will describe a parabolic curve a, b'', c'', d'', e, as represented by the illustration.

Fig. 92.



$z = b\ a\ b'$, the angle of inclination.

V = velocity of the body when at a, which can be dissolved into two component parts—namely,

v = the horizontal velocity with which the body would move from a to b' in the same time it moves from a to b, and

v' = vertical velocity with which the body would move from b' to b in the same time it moves from a to b.

The horizontal velocity v will be uniform, because no force is acting on the body in that direction; but in the vertical direction the force of gravity is constantly acting to draw the body down, so that instead of arriving at b it will arrive at b'', or the force of gravity has drawn the body from b to b''; thus, the vertical velocity is retarded until the body arrives at the vertex c'' of the parabola, from which point that velocity will be accelerated until the body arrives at e with the same velocity as that with which it started from a.

Without the action of the force of gravity the body would continue with a uniform velocity V via b, c, d to e; but the force of gravity causes the body to fall

1 unit of space from b to b''.
4 units of space from c to c''.
9 units of space from d to d''.
16 units of space from e to e'.

The space fallen through is as the square of the time of fall. The divisions on the dotted lines represent the units of space fallen through.

The horizontal constant velocity, $v = V \cos.z.$

The vertical velocity at a, $v' = V \sin.z.$

T = time in seconds occupied by the body from a to the vertex c'' of the parabola, or from c'' to e'.

$$T = \frac{v'}{g} = \frac{V\,sin.z}{g}.$$

t = any time in seconds from a, but less than T.
v'' = vertical variable velocity at the time t.

$$v'' = v' - g\,t = V\,sin.z - g\,t.$$

When $V\,sin.z = g\,T$ the vertical velocity will be 0.

V' = velocity in the direction of motion at any time t.

$$V' = \sqrt{(V\,sin.z - g\,t)^2 + (V\,cos.z)^2}.$$

The height h of the vertex c'' to which the body will ascend is

$$h = \frac{(V\,sin.z)^2}{2\,g}.$$

R = base of the parabola which is generated by the uniform horizontal velocity v and time $2\,T$. R is called the horizontal range.

$$R = 2\,v\,T = 2\,T\,V\,cos.z.$$

$$2\,T = \frac{R}{V\,cos.z} = \frac{2\,V\,sin.z}{g}.$$

$$\text{Horizontal range, } R = \frac{2\,V^2\,sin.z\,cos.z}{g}.$$

With a definite velocity V the horizontal range R is greatest when the angle $z = 45°$, because the product of *sine* and *cosine* is greatest at that angle.

The body will arrive at e with a velocity equal to that with which it started from a, omitting the resistance of the air. The parabola a, b'', c'', d'', e' is described in the same length of time as that in which it would fall from e to e'.

The length L of the parabola a, b'', c'', d'', e' is

$$L = \tfrac{2}{3}\left(R + \sqrt{\frac{R^2}{4} + 9\,h^2}\right).$$

Example. A body is thrown up with a velocity $V = 1000$ feet per second, and at an angle $z = 60°$. Required the time T, velocities v and V', the height h and the horizontal range R?

$$\text{Time } T = \frac{V\,sin.z}{g} = \frac{1000 \times sin.60°}{32.17} = 26.92 \text{ seconds.}$$

The body will complete the parabola in $26.92 \times 2 = 53.84$ seconds.

The velocity V' in $t = 13.46$ seconds, will be

$$V' = \sqrt{(1000 \times sin.60 - 32.17 \times 13.46)^2 + (1000\ cos.60^\circ)^2} = 661.43$$

feet per second.

Horizontal velocity $v = 500$ feet per second.

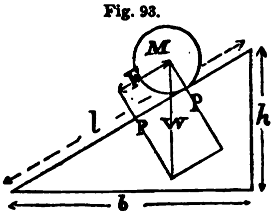
$$\text{The height } h = \frac{(1000\ sin.60)^2}{2 \times 32.17} = 11657 \text{ feet.}$$

Horizontal range $R = 2\ v\ T = 2 \times 500 \times 26.92 = 26920$ feet.

§ 61. MOTION OF BODIES ON INCLINED PLANES.

On the inclined plane l, with the base b and height h, is placed a body or mass M. Draw from the centre of M the vertical line W, to represent the weight of the body M. From the ends of W draw the line P at right angles with the plane l. Draw F parallel with the plane, and complete the parallelogram as shown by Fig. 93.

Fig. 93.



Then, P represents the force with which the body presses against the inclined plane, and F the force which pulls the body in the direction of the plane.

It is well known from geometry that the triangle l, b, h and W, P, F are congruous.

$$W : F = l : h \quad \text{and} \quad W : P = l : b.$$

$$F = \frac{W h}{l} \quad \text{and} \quad P = \frac{W b}{l}.$$

Let T represent the time in seconds from the instant the body is let loose to roll without friction on the plane until it has attained a velocity V. Then, we have

$$F : M = V : T.$$

$$\text{But} \quad F = \frac{W h}{l} \quad \text{and} \quad M = \frac{W}{g}.$$

$$\frac{W h}{l} : \frac{W}{g} = V : T.$$

$$\frac{W h T}{l} = \frac{W V}{g} \quad \text{or} \quad \frac{h T}{l} = \frac{V}{g}.$$

$$\text{Velocity} \quad V = \frac{g h T}{l}, \quad \text{and time} \quad T = \frac{V l}{g h}.$$

Example. The proportional length of the plane is $l=15$ feet in a height $h=3$ feet. Required the velocity of the body after having rolled a time of $T=6$ seconds from rest?

$$V=\frac{32.17\times3\times6}{15}=38.6 \text{ feet per second.}$$

G = acceleratrix, with which the velocity of the body is accelerated on the inclined plane.

$g=32.17$, the acceleratrix when the body falls freely or vertically.

$$\text{Then} \quad G:g=h:l \quad \text{and} \quad G=\frac{g\,h}{l}.$$

The space S in feet, which the body rolls from rest in the time T, will be

$$S=\frac{G\,T^2}{2}=\frac{g\,h\,T^2}{2\,l}=\frac{V\,T}{2}=\frac{V^2}{2\,G}=\frac{V^2\,l}{2\,g\,h}.$$

Example. How far will the body roll in a time $T=25$ seconds on an inclined plane of height $h=15$ feet in $l=100$?

$$S=\frac{32.17\times15\times25}{2\times100}=60.33 \text{ feet.}$$

The time and velocity expressed by space will be

$$T=\sqrt{\frac{2\,S}{G}}=\sqrt{\frac{2\,S\,l}{g\,h}}=\frac{2\,S}{V}.$$

$$V=\frac{2\,S}{T}=\sqrt{2\,G\,S}=\sqrt{\frac{2\,g\,h\,S}{l}}.$$

Example. What velocity will a body attain in rolling a space of $S=64$ feet on an inclined plane of height $h=8$ feet in $l=100$ feet?

$$V=\sqrt{\frac{2\times32.17\times8\times64}{100}}=18.15 \text{ feet per second.}$$

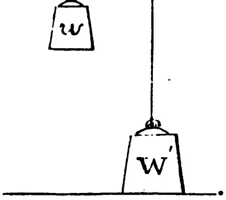
The velocity which a body attains by rolling or sliding without friction or other resistance on an inclined plane is equal to that attained when falling freely a height equal to the difference of level rolled on the plane.

FORMULAS FOR BODIES MOVING FREELY ON INCLINED PLANES UNDER THE ACTION OF GRAVITY.

Velocity in feet per second.	Space in feet of motion.	Time of motion in seconds.
$V=\frac{2S}{T}$. . . . 1	$S=\frac{GT^2}{2}$. . . 4	$T=\sqrt{\frac{2S}{G}}$. . . 7
$V=\sqrt{2GS}$. . 2	$S=\frac{ghT^2}{2l}$. . . 5	$T=\sqrt{\frac{2Sl}{gh}}$. . . 8
$V=\sqrt{\frac{2ghS}{l}}$. . 3	$S=\frac{V^2 l}{2gh}$. . . 6	$T=\frac{2S}{V}$. . . . 9

§ 62. FALLING AND RISING OF BODIES CONNECTED BY A ROPE OVER A PULLEY.

Fig. 94.



The accompanying figure represents two equal weights W and W', hung at each end of a rope over a pulley. As the weights are alike, the one balances the other, and there will be no motion. Hang a third weight w under the weight W, and the system will move with an accelerated velocity until the weight w reaches the ground, when the weight W and W' will continue to move with a uniform velocity equal to that attained at the moment the weight w stopped. It was the force of gravity of the weight w which set the system in motion. Suppose the three weights to be united into one body, and rolled or dragged on a level plane, without friction, by a force equal to the force of gravity of w, that body would move with the same accelerated velocity in equal time as by the force of gravity when hanging on the rope.

Let M denote the mass of the three weights expressed in matts., and F = the force acting to move the system, which is equal to the weight w in pounds.

T = time of action in seconds.

V = velocity in feet per second at the end of the time T.

The fundamental formula for dynamics of matter is

$$M : F = T : V, \qquad \text{and } M\,V = F\,T.$$

$M = \dfrac{F\,T}{V}$. . . . 1	$V = \dfrac{F\,T}{M}$. . . . 3
$F = \dfrac{M\,V}{T}$. . . 2	$T = \dfrac{M\,V}{F}$. . . . 4

Example 1. A mass of $M = 100$ matts., which is 3217 pounds, is acted upon by a force of $F = 150$ pounds (not necessarily weight) for a time of $T = 8$ seconds. Required the velocity V when the force ceases to act?

Formula 3. $V = \dfrac{F\,T}{M} = \dfrac{150 \times 8}{100} = 12$ feet per second.

Example 2. What force $F = ?$ is required to give a mass $M = 360$ matts. a velocity $V = 20$ feet per second, the time of action being $T = 15$ seconds?

Formula 2. $F = \dfrac{M\,V}{T} = \dfrac{360 \times 20}{15} = 48$ pounds.

Example 3. What time $T = ?$ is required to give a mass $M = 580$ matts. a velocity $V = 36$ feet per second, the acting force being $F = 480$ pounds?

Formula 4. $T = \dfrac{M\,V}{F} = \dfrac{580 \times 36}{480} = 43.5$ seconds.

Example 4. A mass of unknown quantities is acted upon by a force of $F = 1680$ pounds for a time of $T = 45$ seconds, in which it attained a velocity of $V = 12$ feet per second. Required the quantity of matter in the mass?

Formula 1. $M = \dfrac{F\,T}{V} = \dfrac{1680 \times 45}{12} = 6300$ matts.,

or about 90 tons, the answer.

Example 5. A mass of $M = 84$ matts. has a velocity of 148 feet per second. What force is required to stop that mass in a time of $T = 6$ seconds?

Formula 2. $F = \dfrac{M\,V}{T} = \dfrac{84 \times 148}{6} = 2072$ pounds.

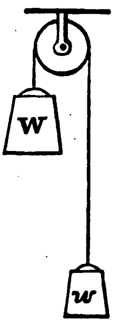
The weight of a body in pounds, multiplied by 0.0310849, will be the mass in matts.

When the mass is expressed by weight the formulas will be

$W = \dfrac{g\,F\,T}{V}$ 5	$V = \dfrac{g\,F\,T}{W}$ 7
$F = \dfrac{W\,V}{g\,T}$ 6	$T = \dfrac{W\,V}{g\,F}$ 8

§ 63. FALLING AND RISING OF BODIES CONNECTED BY A ROPE OVER A PULLEY.

Fig. 95.



The accompanying figure represents two weights W and w hanging on a rope over a pulley. The heavy weight will fall, and draw up the lighter one with an equal velocity. The motive force which sets the system in motion is equal to the difference between the weights, or $F = W - w$. The mass of the two weights is

$$F = \frac{W + w}{g}.$$

The fundamental formula for dynamics of matter, as before described, is

$$M : F = T : V.$$

By inserting the values of F and M in this analogy, we have

$$\frac{W+w}{g} : W - w = T : V,$$

of which the dynamic momentums are

$$V\left(\frac{W+w}{g}\right) = T(W - w).$$

$$\text{Velocity,} \qquad V = \frac{g\,T(W-w)}{W+w}.$$

$$\text{Time,} \qquad T = \frac{V(W+w)}{g(W-w)}.$$

Example. The large $W = 36$ pounds and $w = 30$ pounds. What velocity will the weights attain in a time $T = 3$ seconds from the time the weights start to move?

$$V = \frac{32.17 \times 3(36 - 30)}{36 + 30} = 8.77 \text{ feet per second.}$$

The space S, which the weights will move from the start in the time T, will be $S = \frac{V\,T}{2}$. 6

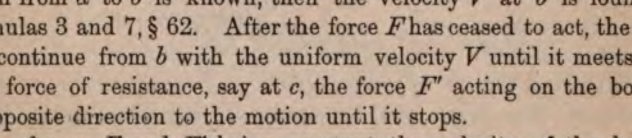
The formulas will hold good when the experiment is made in vacuum, but when the weights W and w are of heavy materials, like that of metals, the difference of fall in air and in vacuum is very small and can be neglected for ordinary purposes. The weight of air displaced by solid iron is only 0.00015 of that of the iron.

When the weights W and w are of light materials, like cork, the difference of fall in air and in vacuum is considerable. The weight of air displaced by cork is 0.005 of that of the cork. That is to say, a pound of cork displaces 33 times as much air as does a pound of iron.

A piece of cork which we may suppose to weigh 1000 pounds would displace 5 pounds of air, and would fall much slower in the air than in vacuum; but let the cork be hung on the rope, Fig. 95, in place of the weight W, and hang another weight of whatever material weighing 5 pounds at w; now let the system move in vacuum under the action of gravity, and the weight W will fall with the same acceleration of velocity as when it falls alone in air.

§ 64. A FORCE ACTING TO MOVE A BODY ON A HORIZONTAL PLANE WITHOUT FRICTION.

Fig. 96.



The body M is supposed to be at rest at a, and a force F is applied to move it from a to b, where the force ceases to act. If the time of action from a to b is known, then the velocity V at b is found by Formulas 3 and 7, § 62. After the force F has ceased to act, the body will continue from b with the uniform velocity V until it meets with some force of resistance, say at c, the force F' acting on the body in an opposite direction to the motion until it stops.

The forces F and F' being constant, the velocity of the body is uniformly accelerated from a to b, and uniformly retarded from c to d, for which the mean velocity in both cases will be $= \frac{V}{2}$.

We have learned (§ 8) that *space* S is the product of time and velocity,

$$\text{or} \quad S = \frac{V\,T}{2} \quad \text{but} \quad V = \frac{F\,T}{M} = \frac{F'\,T'}{M},$$

in which T is the time from a to b, and T' that from c to d.

S = space from a to b, and

s = that from c to d.

$$S = \frac{V\,T}{2} \quad \text{and} \quad s = \frac{V\,T'}{2},$$

$$\text{of which} \qquad V = \frac{2\,S}{T} = \frac{2\,s}{T'} = \frac{F\,T}{M} = \frac{F'\,T'}{M}.$$

The product of either two of these equal members will be alike, or

$$\frac{2\,S}{T} \times \frac{F\,T}{M} = \frac{2\,s}{T'} \times \frac{F'\,T'}{M}, \quad \text{or} \quad \frac{2\,S\,F\,T}{T\,M} = \frac{2\,s\,F'\,T'}{T'\,M},$$

$$\text{of which} \qquad F\,S = F'\,s, \quad \text{or} \quad F : F' = s : S.$$

That is to say, the force F, multiplied by its space of action, is equal to the force F', multiplied by its space s, and the products are momentums of work, which will be explained hereafter.

$F = \frac{F's}{S}$.	1	$S = \frac{F'\,s}{F}$.	3
$F' = \frac{F\,S}{s}$.	2	$s = \frac{F\,S}{F'}$.	4

Example 1. A force $F = 300$ pounds is applied on the body M from a to b, a space of $S = 24$ feet, after which the body meets with a resistance which stops it in a space of $s = 0.5$ feet. Required the force of resistance $F' = ?$

$$\text{Formula 2.} \quad F' = \frac{F\,S}{s} = \frac{300 \times 24}{0.5} = 14400 \text{ pounds, the answer.}$$

The forces F and F' are independent of the weight or mass of the body M, for the reason that the spaces S and s are inverse as the mass, as shown in the proof formulas.

If the applied force F is equal to the weight of the body M, then the body will move on the horizontal plane with an equal accelerated velocity, as if it was falling vertically under the action of gravity; therefore, in falling bodies the acting force is equal to the weight of

the body; which is the same as to say that the weight of a body is the force of attraction between the earth and that body.

The force of a falling body is equal to the resistance it meets with at the end of its fall.

Example 2. A steam-hammer weighing 5 tons falls upon a red-hot mass of iron, in which it sinks $s=1.5$ inches $=0.125$ feet; the height of fall to where the hammer stopped is $S=6.25$ feet, and the force $F=5$ tons. Required the force of the blow of the hammer?

$$\text{Formula 1.} \quad F' = \frac{F S}{s} = \frac{5 \times 6.25}{0.125} = 250 \text{ tons.}$$

It is a popular expression to say "the force of a falling or moving body," or "the force of the blow of a hammer," which is not correct. There is no more force in a moving body than in one at rest; in fact, no force in either case. The force of inertia overcomes the resistance which the moving body strikes, and which resistance is the force which stops the motion of the body.

§ 65. ON DRIVING A NAIL INTO A PIECE OF WOOD.

Fig. 97.

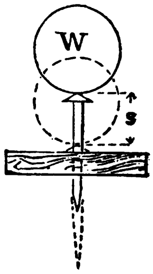


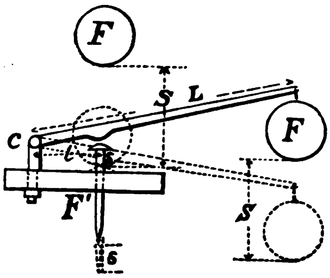
Fig. 97 represents a nail driven through a piece of wood by a weight W resting on the head of the nail. It is supposed that the resistance to the nail in the wood is equal to the weight W, so that the slightest additional force would cause the weight to drive the nail down to its head, as shown by the dotted lines.

In driving a nail into a piece of wood the resistance is not uniform, for the deeper the nail is driven in the greater is the resistance; but the mean force of resistance will always be as the following Formula 2.

Let s denote the space which the nail is driven into the wood by the weight.

Let the same nail be driven into the same piece of wood by the aid of a lever, as represented by Fig. 98. The force F acting on the long lever L, presses on the nail equally to the weight W. The force of resistance F' to the nail in the wood, which is equal to the weight W, Fig. 97, acts on the short lever l. The fulcrum of the lever is at c.

Fig. 98.



The force F with the lever L is adjusted so that it balances the resistance F', acting on the lever l.

From the well-known analogy of levers we have

$$F : F' = l : L.$$

That is to say, the weight F, Fig. 98, is so much smaller than the weight W, Fig. 97, as the lever l is smaller than L.

Let S represent the space which the weight falls in pressing down the nail in the wood, and $s =$ the space the nail was driven in, which is the same as the space s, Fig. 97.

It is well known in geometry that

$$s : S = l : L,$$

and as $\qquad F : F' = l : L,$

we have $\qquad F : F' = s : S, \qquad$ and $F S = F' s$.

$F = \dfrac{F' s}{S}$. . . . 1	$S = \dfrac{F' s}{F}$. . . . 3
$F' = \dfrac{F S}{s}$. . . . 2	$s = \dfrac{F S}{F'}$. . . . 4

These formulas are precisely the same as those in the preceding paragraph; which proves that if the weight F is let to fall from an equal height S directly upon the head of the nail, the latter would be driven into the wood precisely the same distance s as by the aid of the lever and weight F, Fig. 98.

The space of fall S must include the space s of penetration.

§ 66. PILE-DRIVER.

The formulas for pile-drivers are the same as those for force of falling bodies, the principles of action being the same.

NOTATION OF LETTERS.

Fig. 99.

W = weight of the ram in pounds.

S = set of the pile in feet by the blow of the ram.

R = resistance in pounds to the pile in the ground.

h = height in feet which the ram falls, including the set S.

V = velocity in feet per second with which the ram strikes the pile.

$$W : R = S : h.$$

Works $\quad Wh = RS. \quad . \quad . \quad 1$

Weight of ram $\quad W = \frac{RS}{h}. \quad . \quad . \quad 2$

Height of fall $\quad h = \frac{RS}{W}. \quad . \quad . \quad 3$

Resistance to pile $\quad R = \frac{Wh}{S}. \quad . \quad . \quad 4$

Set of pile $\quad S = \frac{Wh}{R}. \quad . \quad . \quad 5$

Striking velocity $\quad V = 8\sqrt{h - S}. \quad . \quad 6$

The ram is raised, either by hand or steam, by a rope over a pulley at the top of the framing.

In using the above formulas some allowance should be made for the work lost by the ram crushing the head of the pile, which may amount to some 25 per cent., or use only 75 per cent. of the actual weight of the ram for W in the formulas.

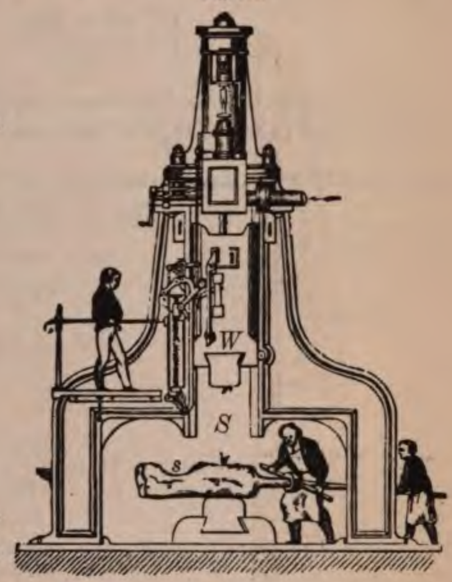
$$\text{Set} \quad S = \frac{0.75\,Wh}{R}.$$

§ 67. STEAM-HAMMERS.

The illustration, Fig. 100, represents a steam-hammer as formerly made by Henry G. Morris of Philadelphia, and is here intended only for illustrating the dynamics of forging. The steam-hammers now made by that gentleman are differently constructed.

FORMULAS FOR STEAM-HAMMERS FALLING UNDER THE ACTION OF GRAVITY.

Fig. 100.



Velocity,

$$V = 8\sqrt{S}.$$

Work,

$$K = W(S+s) = Fs.$$

Force $F = \dfrac{W(S+s)}{S}$.

Horse-power,

$$\text{HP} = \frac{pSn}{33000}.$$

Time $T = \dfrac{\sqrt{S}}{4}$.

$$t = \sqrt{\frac{WS}{16(p-W)}}.$$

NOTATION OF LETTERS.

W = weight of the ram in pounds.

S = stroke of the hammer in feet.

s = depth of penetration by the blow in feet.

V = velocity in feet per second of the blow.

K = work in foot-pounds of the blow.

n = number of blows per minute.

HP = horse-power driving the hammer.

F = mean force of the blow in pounds.

P = steam-pressure in pounds on the top of the piston, when such is used.

T = time of fall of the hammer in seconds.

t = time in seconds required to raise the hammer, the stroke S.

p = steam-pressure in pounds on the under side of the piston.

Example. The weight of a steam-hammer is $W=5000$ pounds, and is lifted $S=4$ feet; the penetration of the blow is $s=0.015$ of a foot. Required the force of the blow?

$$F=\frac{5000(4+0.015)}{0.015}=1,338,333 \text{ pounds},$$

or 600 tons, nearly, the force required.

The steam-pressure p under the piston should be

$$p=\frac{W(S+16\,t^2)}{16\,t^2}.$$

If it is required that the hammer should be lifted in the same length of time as that in which it falls, then make $p=2\,W$.

Formulas for Steam-Hammers falling with aid of Steam on the Top of the Piston.

Velocity	$V=8\sqrt{\frac{S(W+P)}{W}}.$
Work	$K=(W+P)\,(S+s)=Fs.$
Force	$F=\frac{(W+P)\,(S+s)}{s}.$
Horse-power	$\text{HP}=\frac{(W+P)\,S\,n}{33000}.$
Time	$T=\sqrt{\frac{WS}{16\,(W+P)}}.$

$P=$ Steam-pressure on the tops of the piston in pounds (not per square inch).

§ 68. ON FORCE OF MOVING OR FALLING BODIES.

This subject has been treated under different heads in the preceding paragraphs—namely, the Formulas 2 and 6, § 62, give the force in regard to velocity and time, and the Formula 1, § 65, in regard to space. It now remains to explain the subject in regard to velocity and space.

The fundamental analogy of the four elements in dynamics of matter, as before repeated, is

$$M:F=T:V,$$

in which V is the velocity which the force F has given to the mass M in the time T.

Space S is the product of velocity V and time T, but the velocity in the space S, in which the force F started the mass M from rest to the velocity V, is only one-half of the final velocity V, or the uniform velocity of the mass after the force ceased to act. Therefore,

$$S=\frac{V\,T}{2},\quad\text{of which}\quad T=\frac{2S}{V}=\frac{M\,V}{F}.$$

$F=\dfrac{M\,V^2}{2\,S}$. . . .	1	$S=\dfrac{M\,V^2}{2\,F}$. . . .	3
$M=\dfrac{2\,S\,F}{V^2}$. . . .	2	$V=\sqrt{\dfrac{2\,S\,F}{M}}$. . .	4

The formulas will hold equally good whether the force F sets the mass at rest to motion, or brings a moving mass to rest. The mass M must be expressed in matts. of 32.17 pounds each.

When the magnitude of the mass M or body W is expressed in pounds, the formulas will appear as follows:

$F=\dfrac{W\,V^2}{2\,g\,S}$. . . .	5	$S=\dfrac{W\,V^2}{2\,g\,F}$. . . .	7
$W=\dfrac{2\,g\,S\,F}{V^2}$. . . .	6	$V=\sqrt{\dfrac{2\,g\,S\,F}{W}}$. . .	8

Example 1. A rifle-ball of $W=0.02$ pounds was fired with a velocity $V=600$ feet per second into a log of wood, in which it penetrated $S=0.9$ of a foot. Required the force of resistance $F=?$ in the wood?

$$\text{Formula 5.}\quad F=\frac{W\,V^2}{2\,g\,S}=\frac{0.02\times600^2}{2\times32.17\times0.9}=124.34\text{ pounds.}$$

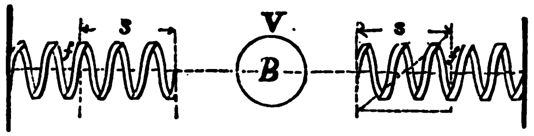
Example 2. A projectile of $W=300$ pounds is fired from a gun $S=10$ feet long with a muzzle velocity of $V=800$ feet per second. Required the mean force of the gunpowder?

$$\text{Formula 5.}\quad F=\frac{300\times800^2}{2\times32.17\times10}=298414\text{ pounds.}$$

§ 69. FORCE OF MOVING BODIES MEASURED BY A SPRING.

The illustration represents a body B moving with a velocity V between two equal spiral springs, the elasticity of which has been measured by experiments that when the spring is compressed the space s, its force of elasticity is f. The force of elasticity of spiral springs generally increases uniformly; that is, when the spring is compressed one-half s, the force will be one-half f, and so on.

Fig. 101.



The height of the dotted triangle may represent the force f for the corresponding space s, from which we see that the mean force in the space s is equal to half the force f.

Let S denote any space the spring may be compressed, and $F' =$ force of the spring corresponding to the space S. Then

$$S : s = F' : f, \qquad S = \frac{s\,F}{f}, \qquad \text{and } F' = \frac{f\,S}{s}.$$

The mean force in the space S will be

$$F = \frac{f\,S}{2\,s}. \quad . \quad . \quad . \quad . \quad . \quad 1$$

$M =$ mass of the body B expressed in matts.

$V =$ velocity of B in feet per second.

$$\text{Mean force,} \quad F = \frac{M\,V^2}{2\,S}. \quad . \quad . \quad . \quad . \quad 2$$

$$\frac{M\,V^2}{2\,S} = \frac{f\,S}{2\,s}, \qquad \text{and } M\,V^2 = \frac{f\,S^2}{s}.$$

$$M = \frac{f\,S^2}{s\,V^2}. \quad . \quad . \quad . \quad 3 \qquad\qquad S = V\sqrt{\frac{s\,M}{f}}. \quad . \quad . \quad 5$$

$$V = \sqrt{\frac{f}{s\,M}}. \quad . \quad . \quad 4 \qquad\qquad \frac{f}{s} = \frac{M\,V^2}{S^2}. \quad . \quad . \quad . \quad . \quad 6$$

Example 1. The mass of the body B is $M=2$ matts. moving against the spring with a velocity $V=5$ feet per second; how much will it compress the spring? when we know that $f=50$ pounds will compress it $s=0.2$ feet?

$$\text{Formula 5.} \quad S=5\sqrt{\frac{0.2\times 2}{50}}=0.447 \text{ feet.}$$

After the spring is compressed by the blow it will push the body back with the same velocity to the other spring, where the same operation would be repeated, and so the body would move between the springs for ever if there was no friction or other resistance to retard and finally stop the motion. It is not necessary that the springs should be alike, only that they should be perfectly elastic within the limit of compression.

Example 2. A body B, weighing $W=50$ pounds, moves against a spring with a velocity $V=25$ feet per second, and compresses it $S=0.5$ feet. Required the mean and ultimate forces of the spring?

$$\text{Force,} \quad F=\frac{50\times 25^2}{2\times 32.17\times 0.5}=1000 \text{ pounds, the mean force}$$

of the spring, and the ultimate force will be double, or 2000 pounds.

§ 70. FORCE OF A BODY FALLING FREELY UPON A SPRING.

Fig. 102.

The weight W falling the space S will compress the spring the space s. After the weight is brought to rest the spring will throw it up again to the same height S, and so the weight will continue to ascend and descend for ever if there is no other force of resistance to retard and finally stop the motion. The weight is acted upon by two opposite forces—namely, the constant force of gravity, which is equal to the weight of the falling body, acts through the space S, and the superior force of the spring acts in the opposite direction in the space s.

Let F denote the mean force of the spring, and f that at the greatest compression s. Then we have, as before proved,

$$W:F=s:S, \quad \text{and } WS=Fs.$$

$$2W:f=s:S, \quad \text{and } 2\,WS=fs.$$

The greatest force of compression $f = 2F$ for ordinary spiral springs.

$W = \frac{Fs}{S}$. 1	$W = \frac{fs}{2S}$. 5
$F = \frac{WS}{s}$. 2	$f = \frac{2WS}{s^2}$. 6
$S = \frac{FS}{s}$. 3	$S = \frac{fs}{2W}$. 7
$s = \frac{WS}{F}$. 4	$s = \frac{2WS}{f}$. 8

In these formulas the spaces S and s can be expressed by any unit of length, and also the weight W and force F by any unit of weight.

Example 1. The weight $W = 4$ pounds falls a space $S = 50$ inches upon a spring, which it compresses $s = 3$ inches. Required the mean force of the spring?

$$\text{Mean force,} \qquad F = \frac{WS}{s} = \frac{4 \times 50}{3} = 66.66 \text{ pounds.}$$

Example 2. It is known by experiments that the spiral spring is compressed $s' = 4.5$ centimètres by a weight $w = 6.3$ kilogrammes resting upon it; from what height S must a weight $W = 3.5$ kilogrammes fall to compress the spring $s = 12.5$ centimètres?

$$s : s' = f : w, \text{ of which } f = \frac{sw}{s'}.$$

I insert this value for f in Formula 7, which will give the required space.

$$S = \frac{ws^2}{2Ws'} = \frac{6.3 \times 12.5^2}{2 \times 3.5 \times 4.5} = 31.25 \text{ centimètres.}$$

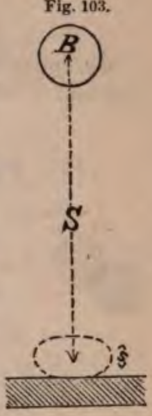
§ 71. ELASTIC BODIES MOVING AGAINST HARD AND IMMOVABLE BODIES.

We have now considered the moving body to be perfectly hard and moved against an elastic spring, but the body may also be perfectly elastic, like that of an india-rubber ball, and strike against a perfectly hard and immovable substance, when the motion of the body will be

the same as when the non-elastic body moved against an elastic spring.

A perfectly elastic body B falling the space S on a perfectly hard and solid base, will be flattened by the blow, or the product of the weight of the body multiplied by its height of fall S, which is equal to the force F of elasticity multiplied by the space s of compression. The force of elasticity will restore the primitive form of the body and raise it to the same height S. The body will so continue to rise and fall for ever if there is no other force to disturb the operation.

Fig. 103.



If the falling body is not perfectly elastic it will not rise to the same height from which it fell, but the height of rise is a measure of the grade of elasticity of the body. If half elastic, it will rise only half the space S, and so on. The elasticity of bodies can thus be found by similar experiments, but it is difficult to find a perfectly non-elastic base to experiment upon, because all bodies in nature are more or less elastic. A cast-iron ball falling upon a massive base of the same material will rebound some, which shows that cast-iron is partly elastic, and even a cobble-stone falling upon a solid rock will indicate some elasticity.

IMPACT OR COLLISION.

§ 72. COLLISION OF PERFECTLY HARD AND NON-ELASTIC BODIES.

A moving hard body striking an immovable one will stop at the collision. Practically, bodies are not perfectly hard nor non-elastic, and when a moving body stops suddenly against an immovable one, the momentum is destroyed by crushing and impressing the parts in contact.

A moving hard body striking one at rest, but free to move, will set the one at rest in motion, so that both bodies will move with equal velocity after the collision.

M = weight of a body of velocity V.

m = weight of another body of velocity v.

It is supposed that the direction of motion passes through the centres of gravity of the operating bodies, and that no momentum is lost in the collision. Although the bodies are denoted by M and m, which means *mass*, they can be expressed by any unit of weight.

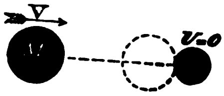
The sum of the momentums of motion of the bodies will be alike before and after the collision, and when the bodies are perfectly hard they will move with a common velocity v' after contact.

$$v'(M+m) = M\,V \pm m\,v.$$

The upper sign + is used when the bodies move in the same direction, and − when in opposite directions.

§ 73. ONLY ONE OF THE HARD BODIES IN MOTION.

Fig. 104.



The body M moves with a velocity V, and strikes the body m at rest, or $v = o$, and consequently $m\,v = o$.

$$v'(M+m) = M\,V.$$

$$v' = \frac{M\,V}{M+m}. \quad . \quad . \quad . \quad 1 \qquad\qquad M = \frac{v'\,m}{V-v'}. \quad . \quad . \quad . \quad 3$$

$$V = \frac{v'(M+m)}{M}. \quad . \quad . \quad 2 \qquad\qquad m = \frac{M(V-v')}{v}. \quad . \quad . \quad 4$$

Example 1. A hard body $M = 24$ pounds moves with a velocity $V = 42$ feet per second, and strikes the body $m = 16$ pounds. Required the common velocity of the bodies after collision?

$$\text{Velocity,} \qquad V' = \frac{24 \times 42}{24+16} = 25.2 \text{ feet per second.}$$

Example 2. A body $m = 500$ pounds, at rest but free to move, is to be set into motion of $v' = 20$ feet per second by a body $M = 80$ pounds. Required the velocity V? of the body M?

$$\text{Velocity,} \qquad V = \frac{20\,(500+80)}{80} = 145 \text{ feet per second, the answer.}$$

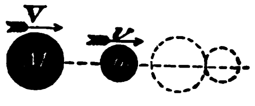
Example 3. A body $M = 250$ pounds moves with a velocity $V = 64$ feet per second. What weight m? must be put in the way so as to reduce the velocity of M from 64 to 5 feet per second?

$$m = \frac{250\,(64-5)}{5} = 3450 \text{ pounds.}$$

It is supposed that the body $m = 3450$ pounds is free to move after the collision.

§ 74. THE TWO HARD BODIES MOVE IN THE SAME DIRECTION.

Fig. 105.



The body M moves fastest until it reaches m, after which they will both move with the common velocity v'.

$$v'(M+m) = M\,V + m\,v.$$

$$v' = \frac{MV+mv}{M+m}. \quad . \quad . \quad 1 \qquad\qquad M = \frac{m(v'-v)}{V-v'}. \quad . \quad . \quad 3$$

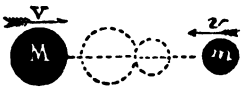
$$V = \frac{v'(M+m)-mv}{M}. \quad . \quad 2 \qquad\qquad m = \frac{M(V-v')}{v'-v}. \quad . \quad . \quad 4$$

Example 4. The body $M=144$ pounds moves with a velocity $V=72$ feet per second, $m=480$ pounds and $v=18$. Required the common velocity v'? after contact?

$$\text{Velocity,} \quad v' = \frac{144\times72+480\times18}{144+480} = 30.15 \text{ feet per second.}$$

§ 75. THE HARD BODIES MOVE IN OPPOSITE DIRECTIONS.

Fig. 106.



The body with the greatest momentum of motion will gain on the other body, and return it with the velocity v'.

$$v'(M+m) = M\,V - m\,v.$$

$$v' = \frac{MV-mv}{M+m}. \quad . \quad . \quad 1 \qquad\qquad M = \frac{m(v'+v)}{V-v'}. \quad . \quad . \quad 3$$

$$V = \frac{v'(M+m)+mv}{M}. \quad . \quad 2 \qquad\qquad m = \frac{M(V-v')}{v'+v}. \quad . \quad . \quad 4$$

Example 5. The body $m=81$ pounds, and moves with a velocity $v=36$ feet per second against $M=27$ pounds. What velocity must the body M have to stop the two bodies in the collision? $v'=0$.

$$\text{Velocity,} \quad V = \frac{0(27+81)+81\times36}{27} = 108 \text{ feet per second.}$$

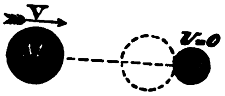
§ 76. COLLISION OF PERFECTLY ELASTIC BODIES.

It has before been explained that when a perfectly elastic body strikes an immovable body, it will rebound with the same velocity as that with which it struck; but if it strikes a movable body, some momentum will be lost by the striking body in giving motion to the one at rest; and in all cases the sum of the momentums before and after collision will be alike, whether either or both bodies are non-, perfectly or partly elastic.

$$\left.\begin{array}{l} V' = \text{velocity of } M \\ v' = \text{velocity of } m \end{array}\right\} \text{after collision.}$$

§ 77. ONLY ONE OF THE ELASTIC BODIES MOVES AND STRIKES THE ONE AT REST.

Fig. 107.



$$V'(M+m) = V(M-m).$$

$$v'(M+m) = 2\,V\,M.$$

$$V' = \frac{V(M-m)}{M+m} \quad . \quad . \quad . \quad 1 \qquad\qquad M = \frac{m(V+V')}{V-V'} \quad . \quad . \quad . \quad 5$$

$$v' = \frac{2\,V\,M}{M+m} \quad . \quad . \quad . \quad 2 \qquad\qquad m = \frac{M(V-V')}{V+V'} \quad . \quad . \quad . \quad 6$$

$$V' = \frac{v'(M-m)}{2\,M} \quad . \quad . \quad . \quad 3 \qquad\qquad M = \frac{m\,v'}{v'-2\,V'} \quad . \quad . \quad . \quad 7$$

$$v' = \frac{2\,V'M}{M+m} \quad . \quad . \quad . \quad 4 \qquad\qquad m = \frac{M(v'-2\,V')}{v'} \quad . \quad . \quad 8$$

Example 6. A body $M = 360$ pounds, moving with an unknown velocity, strikes an elastic body m at rest and of unknown weight. After the collision it is found that the velocity of M is $V' = 42$, and that of m is $v' = 96$ feet per second. Required the weight of the body m?

$$\text{Formula 8.} \quad m = \frac{360(96-2\times 42)}{96} = 90 \text{ pounds.}$$

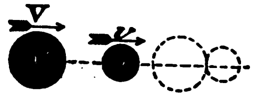
§ 78. TWO PERFECTLY ELASTIC BODIES MOVING IN THE SAME DIRECTION.

The body M moves fastest until it reaches the body m, after which it will move slower or perhaps return.

$$V'(M+m)=V(M-m)+2\,m\,v.$$

$$v'(M+m)=2\,M\,V+v(m-M).$$

Fig. 108.



$$V'=\frac{V(M-m)+2\,m\,v}{M+m}. \quad . \quad 1$$

$$v'=\frac{2\,M\,V+v(m-M)}{M+m}. \quad . \quad 2$$

$$V'=\frac{v'[\,V(M-m)+2\,m\,v]}{2\,M\,V+v(m-M)}. \quad 3$$

$$v'=\frac{V'[2\,M\,V+v(m-M)]}{V(M-m)+2\,m\,v}. \quad 4$$

$$M=\frac{m(\,V'+V-2\,v)}{V-V'}. \quad . \quad 5$$

$$m=\frac{M(2\,V-v-v')}{v'+v}. \quad . \quad 6$$

$$V=\frac{V'(M+m)-2\,m\,v}{M-m}. \quad 7$$

$$v=\frac{v'(M+m)-2\,M\,V}{m-M}. \quad 8$$

Example 7. The body $m=800$ pounds, and moves with a velocity of $v=28$ feet per second. $M=360$ pounds.

It is required to give such a velocity to the body M that when it strikes m all the momentum $M\,V$ will be discharged into m; so that M will be at rest after the collision, and m will move with the two momentums.

$$m\,v'=M\,V+m\,v.$$

Required the velocity V of the body M? and v' of m?

$$\text{Formula 7.}\quad V=\frac{0(360+800)-2\times800\times28}{360-800}=101.8$$

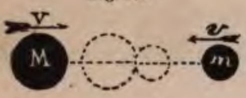
feet per second, the required velocity of the body M.

$$v'=\frac{M\,V+m\,v}{m}=\frac{360\times101.8+800\times28}{800}=38.58 \text{ feet}$$

per second, the velocity of m after the collision.

§ 79. TWO PERFECTLY ELASTIC BODIES MOVE IN OPPOSITE DIRECTIONS.

Fig. 109.



$$V'(M+m) = V(M-m) - 2\,m\,v.$$

$$v'(M+m) = 2\,M\,V - v(m-M).$$

$$V' = \frac{V(M-m) - 2\,m\,v}{M+m} \quad . \quad . \quad 1$$

$$v' = \frac{2\,M\,V - v(m-M)}{M+m} \quad . \quad . \quad 2$$

$$V' = \frac{v'[\,V(M-m) - 2mv]}{2\,M\,V - v(m-M)} \quad . \quad . \quad 3$$

$$v' = \frac{V'[2\,M\,V - v(m-M)]}{V(M-m) - 2\,m\,v} \quad . \quad 4$$

$$M = \frac{m(2vv' + v'V - V'v)}{2VV' + V'v - V'v} \quad . \quad . \quad 5$$

$$m = \frac{M(2\,V\,V' + V'v - v'\,V}{2\,vv' + v'\,V - V'v} \quad . \quad 6$$

$$V = \frac{v(m\,V' - 2\,mv' - M\,V')}{2\,M\,V' + m\,v' - Mv'} \quad . \quad 7$$

$$v = \frac{V(2\,M\,V' + m\,v' - Mv')}{m\,V' - 2\,mv' - M\,V'} \quad . \quad 8$$

Example 8. $M = 24$ kilogrammes, $V = 16$ mètres per second, $m = 10$ kilogrammes, and $v = 36$ mètres per second. Required the velocities V' and v' after the collision?

$$V' = \frac{16\,(24-10) - 2\times10\times36}{24+10} = -17.5 \text{ mètres per second.}$$

The negative sign $-$ means that the body M returns with that velocity.

$$v' = \frac{2\times24\times16 - 36\,(10-24)}{24+10} = +29.88 \text{ mètres per second.}$$

The positive sign + means that the body m continues its course with a velocity reduced from 36 to 29.88 mètres per second.

§ 80. COLLISION OF BODIES OF WHICH EITHER ONE OR BOTH ARE ELASTIC, NON-ELASTIC, OR PARTLY ELASTIC.

Let E denote the grade of elasticity of the body M, and
e = that of the body m.

When a body is perfectly elastic, E or e is equal to 1, and when perfectly hard or non-elastic, E or e is zero or 0. Therefore, when a body is half elastic, E or e is 0.5, and so on.

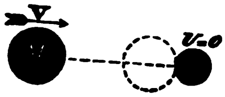
§ 81. ONE BODY IN MOTION AND ONE AT REST.

$$V'(M+m) = V(M-E\,m).$$

$$v'(M+m) = V\,M(1+e).$$

$$\frac{V}{V'}(M-E\,m) = \frac{V\,M(1+e)}{v'}.$$

Fig. 110.



$$V' = \frac{V(M-E\,m)}{M+m} \quad . \quad . \quad 1$$

$$v' = \frac{V\,M(1+e)}{M+m} \quad . \quad . \quad 2$$

$$V' = \frac{v'(M-E\,m)}{M(1+e)} \quad . \quad . \quad 3$$

$$v' = \frac{V'\,M(1+e)}{M-E\,m} \quad . \quad . \quad 4$$

$$M = \frac{E\,m\,v'}{v' - V'(1+e)} \quad . \quad . \quad 5$$

$$m = \frac{M[v' - V'(1+e)]}{E\,v'} \quad . \quad . \quad 6$$

$$V = \frac{V'(M+m)}{M-E\,m} \quad . \quad . \quad 7$$

$$m = M\left[\frac{V}{v'}(1+e) - 1\right] \quad . \quad . \quad 8$$

Example 9. The body $M = 200$ pounds, and elasticity $E = 0.75$, struck the body $m = 150$ pounds, and elasticity $e = 0.25$. After the collision it was found that the velocity of m was $v' = 30$ feet per second. Required the velocity V' of the body M?

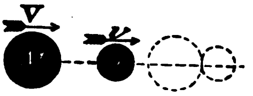
Formula 3. $V' = \dfrac{30\,(200 - 0.75 \times 160)}{200(1+0.25)} = 9.6$ feet per second.

§ 82. THE PARTLY ELASTIC BODIES MOVE IN ONE DIRECTION.

Fig. 111.

$$V'(M+m) = V(M-E\,m) + m\,v(1+E).$$

$$v'(M+m) = M\,V(1+e) + v(m-e\,M).$$



$$V' = \frac{V(M-E\,m)+m\,v(1+E)}{M+m}. \quad 1$$

$$v' = \frac{M\,V(1+E)+v(m-e\,M)}{M+m} \quad 2$$

$$V' = \frac{v'[V(M-E\,m)+m\,v(1+E)]}{M\,V(1+E)+v(m-e\,M)}. \quad 3$$

$$v' = \frac{V'[M\,V(1+E)+v(m-2\,M)]}{V(M-E\,m)+m\,v(1+E)}. \quad 4$$

$$M = \frac{m.[V'v+v'\,V\,E-v'v(1+E)]}{v'v+V'e-V'V(1+E)}. \quad 5$$

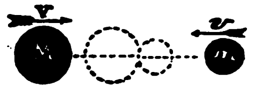
$$m = \frac{M[v'v+V'e-V'V(1+E)]}{V'v+v'\,V\,E-v'\,v(1+E)}. \quad 6$$

$$V = \frac{V'(M+m)}{M-E\,m+m\,v(1+E)}. \quad 7$$

$$v = \frac{v'(M+m)-M\,V(1+E)}{m-e\,M}. \quad 8$$

§ 88. PARTLY ELASTIC BODIES MOVING IN OPPOSITE DIRECTIONS.

Fig. 112.



$$V'(M+m) = V(M-Em) - m\,v(1+E).$$

$$v'(M+m) = M\,V(1+e) - v(m-e\,M).$$

$$V' = \frac{V(M-Em) - m\,v(1+E)}{M+m}. \quad 1$$

$$v' = \frac{M\,V(1+E) - v(m-e\,M)}{M+m}. \quad 2$$

$$V' = \frac{v'[\,V(M-Em) - m\,v(1+E)]}{M\,V(1+E) - v(m-e\,M)}. \quad 3$$

$$v' = \frac{V'[M\,V(1+E) - v(m-e\,M)]}{V(M-Em) - m\,v(1+E)}, \quad 4$$

$$M = \frac{m[\,V'v - v'E - v'v(1+E)]}{V'V(1+E) + V'e - v'\,V}. \quad 5$$

$$m = \frac{M[\,V'V(1+E) + V'e - v'\,V]}{V'v - v'\,E - v'\,v(1+E)}. \quad 6$$

$$V = \frac{V'(M+m)}{M - Em - m\,v(1+E)}. \quad 7$$

$$v = \frac{v'(M+m) - M\,V(1+E)}{m - e\,M}. \quad 8$$

Example 10. A perfectly hard body $M=125$ pounds is moving with a velocity of $V=54$ feet per second against a perfectly elastic body $m=480$ pounds, and $v=24$ feet per second. Required the velocities V' and v' after the collision?

In this case the elasticity of M is $E=0$, and that of m is $e=1$.

Formula 1. $V' = \dfrac{54(125-0\times480) - 480\times24(1+0)}{125+430} = -7.85$ feet per second, the velocity with which the body M will return.

Formula 2. $v' = \dfrac{125\times54(1+0) - 24(480-1\times125)}{125+480} = -2.92$ feet per second, the returning velocity of the body m.

Both the bodies will return in opposite directions after the collision.

DYNAMICS OF MATTER.

§ 84. ON POWER AND WORK APPLIED TO MATTER.

The three physical elements, *force, motion* and *time*, with their combination in *space, power* and *work*, have already been explained—namely,

Elements.		*Functions.*	
Force $=F$ . . .	1	Space, $S=V\,T$ . .	4
Motion $=V$ . . .	2	Power, $P=F\,V$ . .	5
Time $=T$ . . .	3	Work, $K=F\,V\,T$.	6

In the application of this law upon matter, a fourth element, *mass*, enters into the combination which constitutes the dynamics of matter.

The relation between these four elements has already been explained in regard to force and motion, and it now remains to treat them in regard to power and work.

The fundamental analogy in dynamics of matter is

$$M : F = T : V, \quad \text{and } M\,V = F\,T.$$

Multiply both the momentums by the velocity V, and we have the work $K=M\,V^2=F\,V\,T$, which means that the work consumed in giving the mass M the velocity V is $F\,V\,T$, or the product of force, velocity and time.

In the formula for work, $K=F\,V\,T$, V means the mean velocity of the force F in the time T.

When a constant force sets a body (at rest, but free to move) in motion, the velocity at the start is zero or 0, but is accelerated to V in the time T; and the mean velocity in that time is only one-half of the final velocity; therefore, if V means the final or uniform velocity of a moving body after the force has ceased to act upon it, the work stored in the body will be

$$K=M\tfrac{1}{2}\,V^2=F\tfrac{1}{2}\,V\,T.$$

The space in which the body was set in motion is

$$S=\tfrac{1}{2}\,V\,T,$$

which, inserted in the formula for work, will be

$$K=M\tfrac{1}{2}\,V^2=F\,S.$$

A body lifted vertically a space or height S against the force of

gravity, but without regard to velocity and time, the work accomplished is $K = F S$, but the force F is equal to the weight W of the body, and $K = W S$.

Now, let the body fall freely to the same spot from which it was lifted, and the force of gravity will perform an equal work,

$$K = W S.$$

In the falling work the velocity and time might be entirely different from that in the lifting work, but their product will be equal in both cases; that is, the time multiplied by the velocity of the lifting work is equal to the time multiplied by the mean velocity of the falling work.

The space $S = \frac{1}{2} V T$, in which V means the velocity the falling body has attained when striking the ground, and T the time of fall. Insert this value of space in the formulas for work, and

$$K = W \tfrac{1}{2} V T.$$

The time of fall is $$T = \frac{V}{g}.$$

$$K = \frac{W \frac{1}{2} V^2}{g}.$$

Mass, $$M = \frac{W}{g}.$$

Work $K = M \frac{1}{2} V^2$, which was to be proved.

That is to say, the work stored in a moving body is equal to the mass multiplied by half the square of its velocity.

§ 85. DYNAMICS OF MATTER REPRESENTED GEOMETRICALLY.

The combination of the physical elements and functions with matter can be represented by the corresponding geometrical quantities, as explained in § 2, and illustrated by Fig. 113.

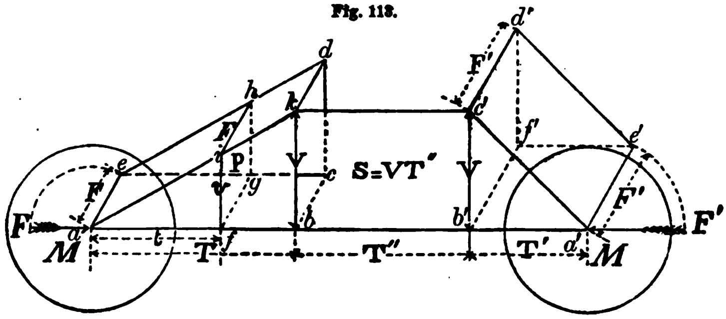
Let the arrow F represent the magnitude and direction of a constant force applied on a body M at rest, but free to move. The body M will then be started and moved with an accelerated velocity as long as the force acts. Let the line T represent the time of action of the force, in which the body moves from a to b, and let the vertical line V represent the velocity attained at b. The force F ceases to act at b, and the body will continue its motion with the uniform velocity V until some other force is applied to change and stop its

motion. At b' the body meets a force F' in opposite direction to its motion which will be stopped at a' in the time T'. Whilst the body moved from b to b', the space $S = V T''$, its momentum of motion was

$$M V = F T = F' T', \quad \text{or} \quad F : F' = T' : T.$$

The illustration is so proportioned that $F' = 2 F$ and $T = 2 T'$.

Fig. 113.



Draw the line $a\ k$, and the area of the triangle a, b, k represents the space

$$S = \frac{V T}{2}$$

which the force F passed through in the time T.

Draw from a the line $a\ e$ at any convenient angle to represent the force F, and complete the wedge or prism a, b, c, d, e. Then the cubic contents of that prism will represent the work K consumed in giving the mass M the velocity V.

$$K = F S = \frac{F V T}{2}.$$

Complete the same operation at the other end of the illustration, and the prism a', b', c', d', e' will represent the work consumed in bringing the mass M to rest.

The cubic contents of the two prisms are alike, or the works

$$K = \frac{F V T}{2} = \frac{F' V T'}{2} = \frac{M V^2}{2}.$$

$$2K = F V T = F' V T' = M V^2.$$

Work is always $K = F V T$, in which the velocity V means the mean velocity in the time T, but in the case of a moving body the velocity V is double the mean velocity of the force which put the body in motion or brought it to rest. Therefore the work of a mov-

ing body is expressed by $K=\frac{F\,V\,T}{2}$, when V means the uniform velocity of the body.

The area of the rectangle a, b, c, e, or the base of the prism, represents the momentum of time $F\,T$, which is equal to the momentum of motion of the moving mass $M\,V$.

The power in operation at any time t is represented by the area of the rectangle f, g, h, i, where the velocity of the mass or force is v, or the section Fv of the prism is the power at the time t. The power varies with the velocity as long as the force acts, after which there is no power in operation. The mean power required to set in motion or bring to rest a body is

$$P=\frac{M\,V^2}{2\,T}=\frac{F\,V}{2}.$$

The work expended in setting a body in motion is restored or re-utilized in bringing the moving body to rest; but the elements of the expended and utilized works need not be alike.

Either one or two of the three physical elements can vary *ad libitum*, but only at the expense of the remaining two or one of them, so that the product of the three elements which constitute the work is alike in both cases.

The two works may be distinguished as follows:

Primitive work, $F\,V\,T=F'\,V\,T'$, the realized work.

The work expended in setting the body in motion is the primitive work, and that restored in bringing the body from motion to rest is the realized work.

The work of steam in a steam-engine is the primitive work, which is equal to the realized work executed by the engine, less that consumed by friction.

The two works can be distinguished in different ways—as *positive* and *negative*; *action* and *reaction*; *primary* and *secondary*; *original* and *final*; *cause* and *result*, etc.

If we call the decomposition of carbonic acid, forming vegetation, a primary work, then the burning of the carbon, forming carbonic acid or generating heat, will be a secondary work. When the generation of heat is considered a primary work, the generation of steam may be termed the secondary, and so on.

Example 1. The mass $M=12$ matts. is set in motion by a force $F=36$ pounds, acting for a time of $T=6$ seconds. Required the work consumed on the mass, and the velocity at the end of six seconds? also the space S in which the force acts, and the mean power of the operation?

$$\text{The velocity}\quad V=\frac{F\,T}{M}=\frac{36\times 6}{12}=18 \text{ feet per second.}$$

The work k will be

$$K=\frac{M\,V^2}{2}=\frac{12\times 18^2}{2}=1944 \text{ foot-pounds.}$$

This work ought to be equal to $\frac{1}{2}F\,V\,T$, or

$$K=\frac{F\,V\,T}{2}=\frac{36\times 18\times 6}{2}=1944 \text{ foot-pounds, the same result.}$$

The space in which the force acted in the six seconds will be

$$S=\frac{V\,T}{2}=\frac{18\times 6}{2}=54 \text{ feet.}$$

The mean power in operation will be

$$P=\frac{F\,V}{2}=\frac{36\times 18}{2}=324 \text{ effects.}$$

Example 2. After the mass $M=12$ matts. received the uniform velocity of $V=18$ feet per second, it encountered a force of resistance $F'=72$ pounds. In what time can that force stop the body?

$$T=\frac{M\,V}{F'}=\frac{12\times 18}{72}=3 \text{ seconds.}$$

The work consumed in stopping the body will be

$$K=\frac{F\,V\,T}{2}=\frac{72\times 18\times 3}{2}=1944 \text{ foot-pounds,}$$

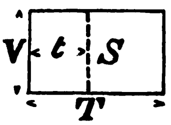
which is the same as the work which set the body in motion.

CALCULUS,

APPLIED TO DYNAMICS.

§ 86. Space.

Fig. 114.



The function *space* is generated by the two elements *velocity* and *time*.

$$S = V\,T.$$

Let the function linear space be represented by the area of a rectangle of which the velocity V represents one side, and the time T the other. Assume the velocity V to be constant, and the space s to vary with the time t, then we have the differentials,

$$\partial s = V\partial t, \text{ of which the velocity } V = \frac{\partial s}{\partial t}.$$

This expression $\frac{\partial s}{\partial t}$ can be inserted for the velocity V in any dynamical formula; for instance,

$$F : M = V : T, \text{ of which } F = \frac{M\,V}{T}.$$

Insert $\frac{\partial s}{\partial t}$ for V in this formula, and we have

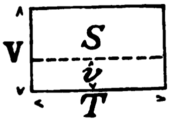
$$F = \frac{M\partial s}{T\partial t}, \text{ or } F\,T\partial t = M\partial s.$$

$$\int F\,T\partial t = \int M\partial s, \text{ or } \frac{F\,T^2}{2} = M\,S,$$

$$\text{of which the space } S = \frac{F\,T^2}{2\,M}.$$

That is to say, a force F acting on a mass M free to move will pass through the space S in the time T, according to the formula.

Fig. 115.



We may also consider the time T to be constant and the velocity variable, as represented by Fig. 115.

$$S = V\,T, \text{ and } \partial S = T\partial v, \text{ of which } T = \frac{\partial s}{\partial v}.$$

This expression $\frac{\partial s}{\partial v}$ can be inserted for the time T in any dynamical formula.

$$F = \frac{MV}{T} = \frac{MV\,\partial v}{\partial s}.$$

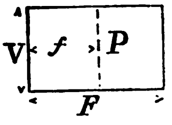
$$F\,\partial s = MV\,\partial v, \quad \text{or} \quad FS = \frac{MV^2}{2},$$

which is the formula for work in dynamics of matter. The work required to bring a mass from rest to motion, or from motion to rest, is

$$K = FS = \frac{MV^2}{2}.$$

§ 87. Power.

Fig. 116.



The function *power* is the product of *force* and *velocity*, which can be represented by a rectangle.

Power $\quad P = FV$ and $\partial P = V\,\partial f,$

when V is constant, or $\quad V = \frac{\partial P}{\partial f}.$

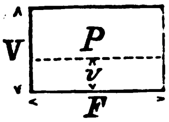
This expression $\frac{\partial P}{\partial f}$ can be inserted for the velocity V in any dynamical formula; for instance,

$$\text{Work} \quad K = FVT = \frac{FT\,\partial P}{\partial f}.$$

$$\int K\,\partial f = \int FT\,\partial P, \quad \text{or} \quad KF = FTP, \quad K = PT;$$

which is to say that work is the product of power and time.

Fig. 117.



We may also assume the force to be constant and the velocity variable, as represented by Fig. 117.

Power $\quad P = FV,$ and $\partial P = F\,\partial v,$

+ $\quad$ of which $\quad F = \frac{\partial P}{\partial v}.$

This expression $\frac{\partial P}{\partial v}$ can be inserted for the force F in any dynamical formula; for instance,

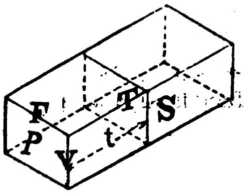
$$\text{Power} \quad P = \frac{FS}{T} = \frac{S\,\partial P}{T\,\partial v}.$$

$$\int P\,T\,\partial v = \int S\,\partial P, \quad \text{or} \quad S = VT.$$

§ 8. Work.

The function *work* is represented by the cubic contents of the parallelopipedon $F\ V\ T$, of which the side F represents force, V velocity and T time.

Fig. 118.



The area of the rectangle $F\ V$ represents power and $V\ T$ space.

$$\text{Work,} \quad K = F\ V\ T = P\ T.$$

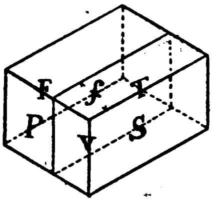
Assume the power P to be constant and the time T variable, we have

$$\partial k = P\,\partial t, \quad \text{and } P = \frac{\partial K}{\partial t}.$$

This expression $\frac{\partial K}{\partial t}$ can be inserted for the power P in any dynamical formula.

Fig. 119 represents $S = V\ T$ to be constant, and the force F variable.

Fig. 119.



$$\text{Work,} \quad K = F\,S, \quad \text{and } \partial K = S\,\partial f.$$

$S = \frac{\partial K}{\partial f}$, which can be inserted for space in any dynamical formula.

$$\text{Work, } K = F\,S, \quad \text{or} \quad \partial K = F\,\partial s,$$

of which $$F = \frac{\partial K}{\partial s}.$$

$\frac{\partial K}{\partial s}$, can be inserted for force in any dynamical formula.

Either or all the elements F, V, T may be variable in the execution of work.

§ 89. POWER AND WORK WITH CONSTANT FORCE AND VARIABLE VELOCITY.

Suppose a constant force F to be applied on a mass M free to move, Fig. 120, it will set the mass in motion with an accelerated velocity. When the force has acted for a time t, the velocity will be v and the power $P = F\,v$.

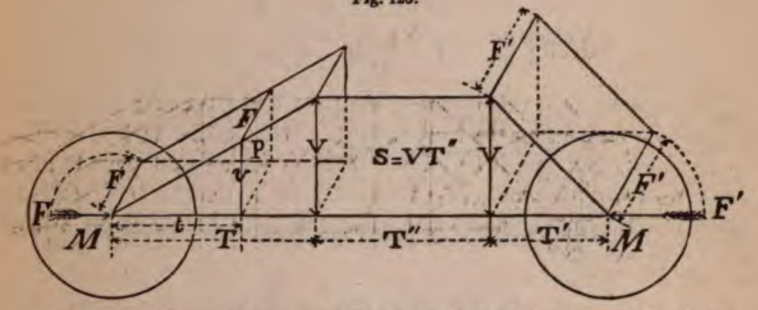
$$F : M = v : t, \text{ of which } t = \frac{M v}{F}, \quad \text{and } \partial t = \frac{M\,\partial v}{F}.$$

The differential work $\partial k = P\,\partial t = F\,v\,\partial t$.

$$\partial k = \frac{F\,v\,M\,\partial v}{F} = M\,v\,\partial v.$$

Work $K = \int M\,v\,\partial v = \frac{M\,V^2}{2}$, which is the well-known formula for work in a moving body.

Fig. 120.



The force F ceases to act at the time T, and the mass M will continue to move with the uniform velocity V and generate the space $S = V\,T''$. There is no work accomplished in the time T'', but the mass will continue to move until some force F' is applied in opposite direction to retard and finally stop the body in the time T'.

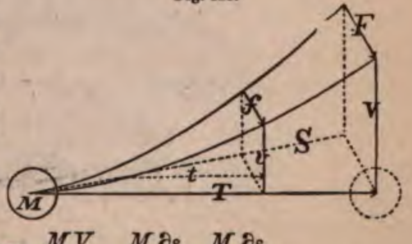
The work which stopped the body is equal to that which set it in motion, and which is represented by the volume of the two prisms,

$$\frac{F\,V\,T}{2} = \frac{F'\,V\,T'}{2}, \quad \text{or} \quad F\,T = F'\,T'.$$

The two momentums of time $F\ T$ and $F'\ T'$ are alike, and each equal to the momentum of motion $M\,V$.

§ 90. FORCE VARIABLE WITH THE TIME OF ACTION.

Fig. 121.



When a force F, acting on a mass M, increases as the time of action, so that $F = C\,T$, C being a constant factor, the relation between time, velocity and space will be as follows:

$$F : M = V : T, \qquad T = \frac{M\,V}{F} = \frac{M\,\partial s}{F\,\partial t} = \frac{M\,\partial s}{C\,T\,\partial t}.$$

$$t\,\partial t=\frac{M\,\partial s}{C},\qquad \int t\,\partial t=\int\frac{M\,\partial s}{C},\qquad \frac{T^3}{3}=\frac{M\,S}{C}.$$

Time $\quad T=\sqrt[3]{\dfrac{3\,M\,S}{C}}$. 1

Space $\quad S=\dfrac{C\,T^3}{3\,M}$. 2

Velocity $\quad V=\dfrac{F\,T}{M}=\dfrac{C\,T^2}{2\,M}$. 3

The velocity is as the square of the time, and the curve is therefore a parabola tangenting the time with its vertex at the start of motion.

Example. Suppose it be known that the force $f=32$ pounds after having acted on the mass $M=50$ matts., for a time of $t=4$ seconds. Required the velocity and space when the force has acted for a time of $T=36$ seconds?

$$f=C\,t,\quad\text{and}\quad C=\frac{f}{t}=\frac{32}{4}=8.$$

Velocity $\quad V=\dfrac{8\times36^2}{2\times50}=103.7$ feet per second.

Space $\quad S=\dfrac{8\times36^3}{3\times50}=2488.3$ feet.

The area bounded within $V\,T$ represents the space passed through; the area of the section $f\,t$ represents the power in operation. The area of the base triangle represents the momentum of time $\frac{1}{2}\,F\,T=M\,V$, the momentum of motion, and the cubic content of the figure represents the work done by the force F, which is $K=\dfrac{M\,V^2}{2}$.

$$\partial K=f\,v\,\partial t=\frac{2\,M\,v\,\partial v}{2}=M\,v\,\partial v.$$

The force $\quad f=C\,t.$

$$\partial K=C\,v\,t\,\partial t=M\,v\,\partial v.$$

$$\int t\,\partial t=\int\frac{M\,\partial v}{C}=\frac{t^2}{2}=\frac{M\,v}{C}.$$

Time $\quad T=\sqrt{\dfrac{2\,M\,V}{C}}$. 4

The same time is obtained from Formula 2.

When the mass is expressed by weight W, the formulas will be

Time $\quad T=\sqrt[3]{\dfrac{3WS}{gC}}$. 5

Velocity $\quad V=\dfrac{gCT^2}{2W}$. 6

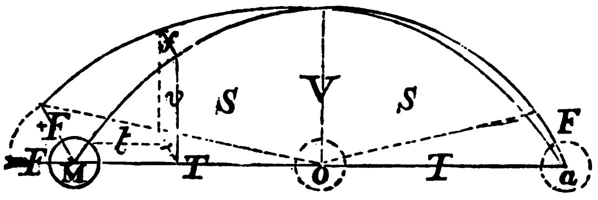
Space $\quad S=\dfrac{gCT^3}{3W}$. 7

Time $\quad T=\sqrt{\dfrac{2MV}{gC}}$. 8

§ 91. FORCE VARIABLE INVERSELY AS THE TIME OF ACTION.

A force F is applied to move a mass M toward o, and the force diminishes as the time of action; so that in a time T the force f is reduced to o. Force $f=C(T-t)$.

Fig. 122.



$$F:M=v:t,\qquad \partial v=\frac{F\partial t}{M}=\frac{C(T-t)\partial t}{M}.$$

$$v=\frac{C}{M}\int T\partial t-t\,\partial t.$$

Velocity $\quad v=\dfrac{C}{M}\left(Tt-\dfrac{t^2}{2}\right)$. . . . 1

$$\frac{\partial s}{\partial t}=\frac{C}{M}\left(Tt-\frac{t^2}{2}\right),\qquad \partial s=\frac{C}{M}\left(Tt-\frac{t^2}{2}\right)\partial t.$$

Space $\quad s=\dfrac{C}{M}\left(\dfrac{Tt^2}{2}-\dfrac{t^3}{6}\right)$. . . . 2

When the formulas are integrated for the whole time T, we have

Velocity $V=\frac{C\,T^2}{2\,M}$. 3

Time $T=\sqrt{\frac{3\,S\,M}{C}}$. 4

Space $S=\frac{C\,T^3}{3\,M}$. 5

These formulas are the same as those in the preceding paragraph. When the body M passes the centre o, the force f becomes negative and stops the motion at a; which operation is accomplished in equal length of time T and space S as that in which it was set in motion from rest to the velocity V at o.

Example. A force $F=240$ pounds sets a body $M=15$ matts. in motion, so that in a time $T=30$ seconds of action the force is reduced to 0. Required the velocity of the body when arriving at 0, and what space it has passed through?

$$F=C\,T \text{ and } C=\frac{F}{T}=\frac{240}{30}=80.$$

Formula 3. Velocity $V=\frac{80\times 30^2}{2\times 15}=2400$ feet per second.

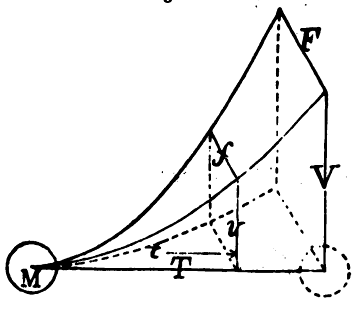
Formula 5. Space $S=\frac{80\times 30^3}{3\times 15}=1600$ feet.

The body arrived at 0 with a velocity of 2400 feet per second, and it will continue to move with that velocity until some force is applied to change it; but if the force f continue to act negatively in the same ratio, the body will be brought to rest at a, 1600 feet from 0.

§ 92. FORCE VARIABLE AS THE SPACE OF ACTION.

Assume $F=C\,S$, in which C is a constant factor.

Fig. 123.



$$F:M=V:T.$$

$$T=\frac{M\,V}{F}=\frac{M\,V}{C\,S}=\frac{M\,\partial s}{C\,s\,\partial t}.$$

$$t\,\partial t=\frac{M\,\partial s}{C\,s}.\quad \int t\,\partial t=\int\frac{M\,\partial s}{C\,s}.$$

$$\frac{T^2}{2}=\frac{M}{C}\,hyp.log.S.$$

Time $T = \sqrt{\frac{2M}{C} hyp.log.S.}$ 1

Velocity $V = \frac{FT}{M} = \frac{CST}{M} = \frac{CS\partial S}{M\partial V}.$

$$\int V\partial V = \int \frac{CS\partial S}{M}.$$

Velocity $V = S\sqrt{\frac{C}{M}}.$ 2

$Hyp.log.S = \frac{CT^2}{2M}.$ 3

Example. The force $F = 3$ pounds at a distance $s = 1$ foot, and the mass $M = 8$ matts. Required the time and velocity of the body at a distance $S = 16$ feet?

$$C = \frac{F}{S} = \frac{3}{1} = 3.$$

Time, $T = \sqrt{\frac{2 \times 8}{3} hyp.log. 16} = 3.845$ seconds.

Velocity, $V = 16\sqrt{\frac{3}{8}} = 9.8$ feet per second.

When the mass is expressed by weight W, we have

Time, $T = \sqrt{\frac{2W}{gC} hyp.log.S.}$ . . . 4

Velocity, $V = S\sqrt{\frac{gC}{W}}.$ 5

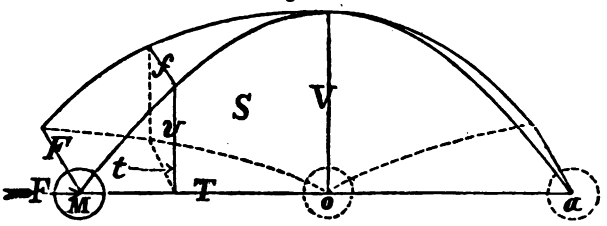
$Hyp.log.S = \frac{gCT^2}{2W}.$ 6

§ 93. FORCE VARIABLE INVERSELY AS THE SPACE OF ACTION.

A force F is applied to move a mass M toward o and diminishes as the space S, so that at o the force is reduced to nothing.

Force $f = C(S - s)$.

Fig. 124.



$$F : M = V : T, \qquad \partial v = \frac{F\,\partial t}{M} = \frac{C(S-s)\partial t}{M}.$$

$$\partial t = \frac{\partial s}{v}, \qquad \text{then } \partial v = \frac{C}{Mv}(S-s)\partial s.$$

$$\int v\,\partial v = \int \frac{C}{M}(S-s)\partial s, \qquad \frac{v^2}{2} = \frac{C}{M}\left(S\,s - \frac{s^2}{2}\right).$$

Velocity, $\quad v = \sqrt{\frac{2C}{M}\left(S\,s - \frac{s^2}{2}\right)}.$ 1

When the velocity is integrated through the whole space S, we have $s = S$ and the

Velocity, $\quad V = S\sqrt{\frac{C}{M}}$. 2

The time of action through the space S will be

$$\frac{\partial s}{\partial t} = S\sqrt{\frac{C}{M}}, \qquad \text{and } \partial t = \frac{\partial s}{S}\sqrt{\frac{n}{C}}.$$

$$t = \int \frac{\partial s}{S}\sqrt{\frac{M}{C}} = \frac{s}{S}\sqrt{\frac{M}{C}}.$$

Time, $\quad T = \sqrt{\frac{M}{C}}$. 3

That is to say, the time required to move the body to o is independent of the distance S as long as $F = C\,S$.

Example. A mass $M=8$ matts. is acted upon by a force $F=48$ pounds at a distance $S=16$ feet. Required the time and velocity at 0?

$$F=C\,S, \qquad \text{and } C=\frac{F}{S}=\frac{48}{16}=3.$$

Velocity, $V=16\sqrt{\frac{3}{8}}=9.8$ feet per second.

Time, $T=\sqrt{\frac{8}{3}}=1.633$ seconds.

When the mass is expressed by weight W, we have

$$\text{Velocity,} \qquad V=S\sqrt{\frac{g\,C}{W}}. \quad . \quad . \quad . \quad . \quad . \quad 4$$

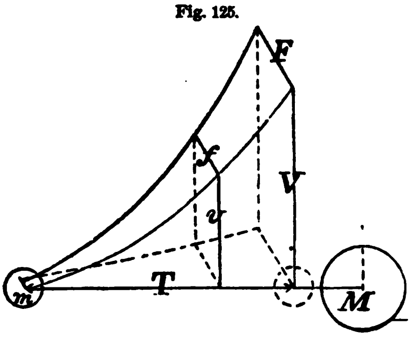
$$\text{Time,} \qquad T=\sqrt{\frac{W}{g\,C}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 5$$

After the body has passed the centre o the force F is negative and stops the body at an equal distance $-S$.

§ 94. FORCE VARIABLE INVERSELY AS THE SQUARE OF THE DISTANCE FROM THE FORCE OF ACTION TO A GIVEN POINT.

The force of attraction between bodies is inversely as the square of their distance apart.

Fig. 125.



Let M and m denote the masses of two bodies expressed in matts, and S their distance apart in feet. The force of attraction between them is

$$F=\frac{M\,m}{\varphi\,S^2}. \quad . \quad . \quad 18$$

$\varphi=28693080$, the coefficient of attraction, § 51.

Suppose the mass M to remain stationary and draw the mass m to it. $s=$ distance moved by m.

$$F:M=V:T, \qquad T=\frac{M\,V}{F}=\frac{M\,V\,\varphi(S-s)^2}{M\,m}.$$

$$T=\frac{\varphi(S-s)^2 V}{m}=\frac{\varphi(S-s)^2}{m}\cdot\frac{\partial s}{\partial T}.$$

$$T\,\partial T=\frac{\varphi}{m}(S^2-2\,S\,s+s^2)\partial s.$$

$$\frac{T^2}{2}=\frac{\varphi}{m}\left(S^2\,s-S\,s^2+\frac{S^3}{3}\right).$$

Time, $$T=\sqrt{\frac{2\,\varphi}{m}\left(S^2\,s-S\,s^2+\frac{s^3}{3}\right)}. \quad . \quad . \quad . \quad 6$$

This is the time in which the mass m will move a space s.

Velocity $$V=\frac{F\,T}{M}=\frac{M\,m\,T}{m\,\varphi(S--s)^2}=\frac{M\,T}{\varphi(S-s^2)}.$$

$$T=\frac{\partial s}{\partial V},\quad \text{then}\quad V=\frac{M\,\partial s}{\varphi(S-s)^2\,\partial V}.$$

$$V\,\partial V=\frac{M\,\partial s}{\varphi(S-s)^2},\qquad \frac{V^2}{2}=\frac{M}{\varphi(S-s)}.$$

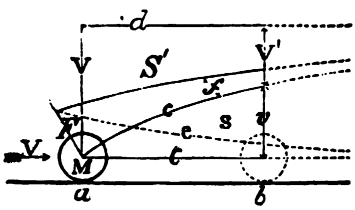
Velocity $$V=\sqrt{\frac{2\,M}{\varphi(S-s)}}. \quad . \quad . \quad . \quad . \quad 7$$

That is to say, the velocity is inversely as the square root of the distance between the attracting bodies.

§ 95. RESISTANCE OF AIR TO A MOVING BODY.

A mass M rolling without friction on a level plane $a\,b$ arrives with a velocity V at a, where it is left to work its own way against the resistance of the air.

Fig. 126.



The force of resistance to a body moving in any perfect fluid is as the square of the velocity; wherefore the resistance at a can be represented by $F=C\,V^2$, of which C is a constant to be determined hereafter.

When the body has moved from a to b in the time t the resistance of the air has reduced the velocity

from V to V', or $V' = V-v$. The force of resistance of the air at b can therefore be represented by $f = C(V-v)^2$.

$$F : M = V : T, \text{ and } T = \frac{MV}{F}.$$

$$\delta t = \frac{M\,\delta v}{f} = \frac{M\,\delta v}{C(V-v)^2}.$$

$$t = \int \frac{M\,\delta v}{C(V-v)^2} = \frac{M}{C(V-v)} - \frac{M}{CV}. \quad . \quad . \quad . \quad 1$$

This formula gives the time in which the velocity V is reduced to $V' = (V-v)$.

The body's motion will be stopped when $V' = 0$ or when $v = V$, which should be when

$$t = \frac{M}{C(V-V')} - \frac{M}{CV} = \infty - \frac{M}{CV}.$$

That is to say, the resistance of the air requires an infinite length of time to stop the body, or, more correctly, the body will never be stopped by that means alone, but the velocity will be reduced so that no motion could be perceptible in days or years.

By solving the Formula 1 we find the velocity

$$v = \frac{CtV^2}{M + CtV}. \quad . \quad . \quad . \quad . \quad 2$$

$$V' = (V-v) = V - \frac{CtV^2}{M + CtV}. \quad . \quad . \quad . \quad 3$$

Call $S =$ the linear space the body would move through with the velocity V in the time t without the resistance of air, or $S = Vt$.

$S' =$ the actual distance moved in the time t against the resistance of the air.

$s =$ the retarded distance in the time t, or $s = S - S'$.

From the Formula 1 we obtain

$$v\left(\frac{M}{V} + Ct\right) = CVt = \frac{\delta s}{\delta t}\left(\frac{M}{V} + Ct\right)$$

$$\delta s = \frac{CVt\,\delta t}{\frac{M}{V} + Ct}.$$

$$S = Vt - \frac{M}{C}\left[hyp.log.(M + CVt) - hyp.log.M\right]. \quad . \quad . \quad 4$$

This linear space is represented by the area of the figure bounded within the lines $c\ t\ v$.

The space S' which the body actually moves in the time t will be

$$S' = \frac{M}{C}\left[hyp.log.(M + C\,V t) - hyp.log.M\right] \quad . \quad . \quad . \quad . \quad . \quad 5$$

This linear space is represented by the area of the figure bounded within the lines $V\,d\,V'\ c$.

The linear space S which the body would have moved through with the constant velocity V in the time T is represented by the area of the rectangle Vt in the figure.

The area of the base bounded within the lines $F t\, e$ represents the momentum of the resistance, which is equal to the momentum Mv.

The volume of the figure bounded within the lines $F v\, t$ represents the work done by the resistance of the air, and which is equal to the work $\frac{1}{2}Mv^2$.

The time required for the body to move through the space S' is found as follows:

$$\text{Call} \qquad X = M + C\,V t. \quad . \quad . \quad . \quad . \quad 6$$

$$Hyp.log.X = \frac{C\,S'}{M} + hyp.log.M. \quad . \quad . \quad . \quad 7$$

$$\text{Time} \qquad t = \frac{X - M}{C\,V}. \quad . \quad . \quad . \quad . \quad . \quad 8$$

The common logarithm multiplied by 2.30258509 is the hyperbolic logarithm.

When the time t occupied by the body in passing through the space S' is correctly known, the initial velocity V will be

$$V = \frac{X - M}{Ct}.$$

§ 96. Coefficient C.

It now remains to find the coefficient C for the resistance of the air.

The resistance is equal to the weight of a column of air with a base equal to the projecting area of the moving body, and a height equal to that from which a body falls and attains the same velocity as that of the resistance.

A cubic foot of air of temperature 60° Fahr. and under a pressure of 30 inches of mercury may be assumed to weigh 530 grains.

A = area of resistance in square feet of the moving body.

h = height in feet of the column of air.

The force f of the resistance of the air will then be in pounds.

$$f = \frac{530\,A\,h}{7000}. \qquad . \quad . \quad . \quad . \quad . \quad 9$$

The height $h = \dfrac{V^2}{2g}$.

$$f = \frac{530\,A\,V^2}{7000 \times 2g}. \qquad . \quad . \quad . \quad . \quad 10$$

Of which the coefficient $C = \dfrac{530\,A}{7000 \times 2g} = \dfrac{A}{849.7734}$. . . . 11

When the area is expressed in square inches a, we have the coefficient

$$C = \frac{a}{122367}. \qquad . \quad . \quad . \quad . \quad 12$$

A or a means a flat surface at right angle to the direction of motion. For a cylinder moving with its convex side to the motion the area of resistance is one-half of the projecting area. When the moving body is spherical, the area of resistance is one-quarter of the projecting area.

D = diameter in feet, and d = diameter in inches of a moving sphere.

$$A = 0.19635\,D^2, \quad \text{and} \quad a = 0.19635\,d^2.$$

Example. A cast-iron ball of $d = 8$ inches in diameter, weighing $W = 69.88$ pounds or $M = 2.1$ matts, is fired from a gun with a velocity $V = 1000$ feet per second. Required the time t in which the ball will reach a target at $S' = 1500$ feet horizontal distance from the muzzle of the gun?

$$a = 0.19635 \times 8^2 = 12.5664 \text{ square inches.}$$

Coefficient $C = \dfrac{12.5664}{122367} = \dfrac{1}{9737.65}$.

$$Hyp.log. X = \frac{1500}{9737.65 \times 2.1} + hyp.log.\ 2.1 = 0.815253$$

$Hyp.log.$ 2.37 = 0.815253, and $X - M = 0.27$.

Time $t = \dfrac{9737.65 \times 0.27}{1000} = 2.629$ seconds.

In this time the ball would fall

$$h = \frac{32.17 \times 2.629^2}{2} = 111.17 \text{ feet.}$$

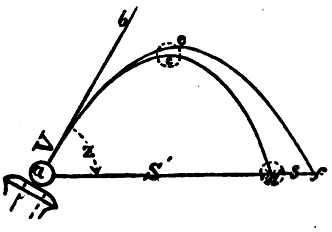
Then, in order to hit the mark on the target, the centre line of the gun must be pointed to 111.17 feet above that mark, or the gun must be elevated to an angle

$$tan.\Phi = \frac{111.17}{1500} = tan. 4^\circ\ 15'.$$

§ 97. RESISTANCE OF THE AIR WHEN THE BODY DESCRIBES A PARABOLA.

When the gun is elevated to an angle z and the ball describes a parabolic curve $a\ c\ d$, the horizontal velocity $v = V\ cos.z$, which inserted for V in the preceding formulas, will make them answer for this case also. The vertical action of resistance in the ascent is counteracted in the descent, so that only the horizontal resistance of the air need be considered in the operation.

Fig. 127.



Without resistance of the air the ball would describe the parabola $a\ e\ f$, and make the horizontal range $S = S' + s$, but the resistance diminishes that space by s or to $S' = S - s$.

MEAN FORCE.

A variable force acting on a body free to move can be converted into a mean force.

The mean force in time is the mean force of momentum.

The mean force in space is the mean force of work.

The mean force in time is at the centre of gravity of the momentum area of the base.

The mean force in space is at the centre of gravity of the work volume.

§ 98. MEAN FORCE Φ IN TIME T. Figs. 121 and 122.

When a force varies directly or inversely as the time, the mean force in the time T is $\Phi = \frac{1}{2} F$, and in the time t, $\Phi = \frac{1}{2}(F+f)$.

The mean force Φ acting on a body in the time T will produce the same velocity as that of the variable force in the same time.

The momentums of time, $\Phi\ T = \frac{1}{2}\ F\ T = M\ V$, the momentum of motion.

§ 99. MEAN FORCE Φ IN SPACE S. Fig. 123.

When a force varies directly as the time of action the mean force in the space S is $\Phi = \frac{2}{3}\ F$, and in the space s $\Phi = \frac{2}{3}(F+f)$.

The mean force acting through the space S, will store the same work in the body as that of the variable force in the same space.

§ 100. MEAN FORCE Φ IN SPACE S. Fig. 122.

When the force varies inversely as the time, the mean force Φ in the space S will be as follows:

Work, $\quad \Phi\ S = \frac{1}{2}\ M\ V^2.$

Insert Formula 5, § 91, for S and Formula 3 for V.

Work, $\quad \Phi \dfrac{C\ T^3}{3\ M} = \dfrac{M\ C^2 T^4}{8\ M^2}.$

$$\Phi = \frac{3\ C\ T}{8} = \frac{3\ F}{8}.$$

Mean force, $\quad \Phi = \frac{3}{8}\ F.$

That is to say, when the force varies either directly or inversely as the time of action, the mean force in the space passed through is $\Phi = \frac{3}{8}\ F.$

Work, $\quad \Phi\ S = \frac{3}{8}\ F\ S = \frac{1}{2}\ M\ V^2.$

§ 101. MEAN FORCE Φ IN SPACE S. Figs. 123 and 124.

When a force varies directly or inversely as the space passed through the mean force in the space S is $\Phi = \frac{1}{2}\ F$, and in the space s, $\Phi = \frac{1}{2}(F+f)$.

Work, $\quad \Phi\ S = \frac{1}{2}\ F\ S = \frac{1}{2}(F+f)s = \frac{1}{2}\ M\ V^2.$

§ 102. MEAN FORCE Φ IN TIME T. Fig. 123.

When a force varies directly as the space, the mean force Φ in the time T is found as follows:

Momentum of time $\Phi\ T = M\ V$, momentum of motion.

Insert Formula 1, § 91, for T and Formula 2 for V.

$$\Phi\sqrt{\frac{2M}{C}\,hyp.log.S} = MS\sqrt{\frac{C}{M}}.$$

$$F = CS.$$

Mean force, $$\Phi = \frac{F}{hyp.log.S}.$$

§ 103. MEAN FORCE Φ IN TIME T. Fig. 124.

When the force varies inversely as the space, the mean force Φ in the time T is found as follows:

Momentum of time $\Phi\,T = M\,V$, momentum of motion.

Insert Formula 3, § 92, for T and Formula 2 for V.

$$\Phi\sqrt{\frac{M}{C}} = MS\sqrt{\frac{C}{M}}.$$

Mean force, $\Phi = CS.$

§ 104. VIS-VIVA.

Vis-viva, literally translated, means living force; the term is used to denote double the work stored in a moving body, or $M\,V^2$.

Force of any kind is an element, and should not denote work, which is a function. The term *Vis-viva* conveys an idea which has often been entertained—namely, that a dead body can possess a virtue of life; which erroneous notion has caused much discordance in the elucidation of dynamics.

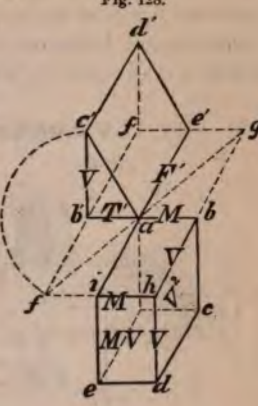
Any change of motion in a body, whether from rest to motion or from motion to rest, requires the same kind of force—namely, that opposing the force of inertia, which should not be termed living force.

In order to form a clear conception of this mysterious *Vis-viva*, we shall bring it bodily in sight, so that we can look at it.

In the accompanying illustration, Fig. 128, the prism a', b', c', d', e' is the same as that in Fig. 113, which represented the work consumed in bringing the moving mass M of velocity V to rest. In the base of this prism we have two elements, force F' and time T', the product of which makes the rectangle a', b', f', e' the momentum of time. The height of the prism represents the velocity V of the mass M.

Continue the line a' b' to f, and make b' $f = b'$ f'. Draw through a' the diagonal f g and complete the parallelogram f, f', g, h; continue the line b' a' to b, then the line a' b represents the mass M of the moving body, and the line a' $i = b$ h represents the velocity V. The parallelogram a', b, h, i is then the momentum of motion M V, which is equal to the parallelogram a', b', f', e', the momentum of time, or F' T' $= M$ V.

Fig. 128.



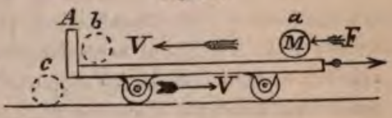
Draw the line h $d = V$ and complete the parallelopipedon a', b, c, d, e, i, which will then represent M V^2, the so-called *Vis-viva*, which is a function of matter and motion.

The cubic content of the parallelopipedon M V^2 is double that of the prism which represents the work of bringing the mass M of velocity V to rest.

§ 105. VIS-VIVA OR LIVING FORCE IN A BODY AT REST.

Fig. 129 represents a platform car attached to a locomotive running on a railway track with a velocity V. On the fore end of the platform is placed a body or mass M moving with the car, but is stationary in relation to the moving system. A force F is applied on the body M to move it backward until it strikes the obstruction A with a velocity V equal to the foreward motion of the car. The body will then strike the obstruction A with a living force or *Vis-viva* M V^2.

Fig. 129.



Now remove the obstruction A and place the body M in its original position at a, and apply the same force F to move the body backward, so that it will attain the same velocity V at b; as there is no obstruction at A, the body will fall down on the track, where it will lie perfectly still, but full of living force, or the *Vis-viva* stored into it by the force F on the platform car.

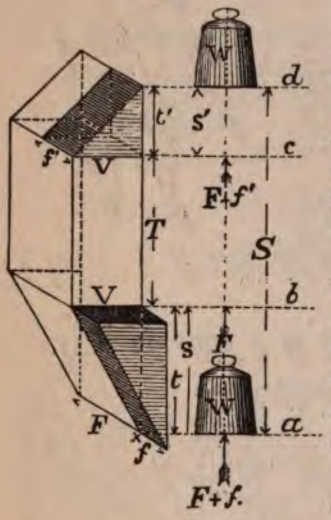
In the first case the *Vis-viva* of the body was discharged by the obstruction A, but in the second case the obstruction was removed, and no discharge of *Vis-viva* has taken place.

This illustration is intended to prove that there is no more *Vis-viva*

in a body when in motion than when at rest. The body M was originally set in motion with the car until it met the resistance F, which brought the body to rest at b, but the obstruction A set it in motion again with the car, and in the last case, when the obstruction was removed, there was no force to set it in motion with the car, for which reason the body remained at rest on the track.

§ 106. ON RAISING OR LIFTING A BODY VERTICALLY.

Fig. 130.



The illustration represents a weight W to be raised from a to d by a force $F = W$, or the force will just balance the weight. The force is represented to push the weight upward, the effect of which is the same as if applied in the eye-bolt to lift the weight; but the illustration is clearer by representing the force to push.

The force F only balances the weight W, and can thus not produce motion, for which an additional force f is required. The additional force f is applied only in the height a, b, or space s in the time t, in which it produces the velocity V, or

$$V = \frac{f t g}{W}.$$

The acceleratrix g is inserted to convert the weight W into mass. After the force f has ceased to act, the weight W will continue with the force F in a uniform ascending velocity until some other force is applied to retard and finally stop the motion. When the weight has reached the height c, the force F is diminished by a force f', which portion of the force of gravity of the weight W acts to retard the motion and bring the weight to rest at d. The force f' acted only in the space s' and time t', so that the momentums of time $f t = f' t'$, or the momentum of time consumed in setting the weight in motion from a to b, is re-utilized from c to d. There is no relation between the forces f and f', which are entirely dependent upon their respective times of action.

In the diagram of work the shaded prism $f\,V t$ represents the primitive work consumed in setting the weight in motion from a to b, which is equal to the shaded prism $f'\,V t'$, or the realized work of inertia which is contributed to that of raising the weight from c to d.

Therefore, the work consumed in raising a weight W a vertical space S is equal to the product of the weight and space, or

$$\text{Work} \quad K = WS.$$

It appears in this formula that work is independent of time and velocity, because the work will be the same for whatever time occupied in raising the weight, and also with whatever velocity it is raised; but the space S cannot be generated without its constituent elements —time and velocity; and either one of these two elements can vary only at the expense of the other, so that their product gives the space, or $S = V T$.

No work can be accomplished without either one of its constituent elements $F\ V\ T$. Either one or two of the elements can vary *ad libitum*, but only at the expense of the remaining two or one.

The forces f and f' correspond with the forces F and F', Fig. 113.

Example. In reference to the illustration Fig. 130, we may assume the force F or weight $W = 50$ pounds, and the additional for $f = 5$ pounds acting in a time of $t = 1$ second. Required the velocity V?

$$V = \frac{f\,t\,g}{W} = \frac{5 \times 1 \times 32.17}{50} = 3.217 \text{ feet per second.}$$

$$\text{The space} \quad s = \frac{V\,t}{2} = \frac{3.217 \times 1}{2} = 1.6085 \text{ feet.}$$

The body is now ascending with the uniform velocity $V = 3.217$ feet per second for a time, say $T = 3$ seconds, which will make a space of $3 \times 3.217 = 9.651$ feet, when the force F is diminished with $f' = 10$ pounds. Required the time t', in which the body will be stopped?

$$\text{Time} \quad t' = \frac{W\,V}{f'\,g} = \frac{50 \times 3.217}{10 \times 32.17} = 0.5 \text{ of a second.}$$

$$\text{The space} \quad s' = \frac{V\,t'}{2} = \frac{3.217 \times 0.5}{2} = 0.804 \text{ of a foot.}$$

The whole operation of lifting the weight was accomplished in

Time.	**Space.**
$t = 1$ second.	$s = 1.6085$ feet.
$T = 3$ seconds.	$b\,c = 3.651$ feet.
$t' = 0.5$ seconds.	$s' = 0.804$ feet.
$t + T + t' = 4.5$ seconds.	$S = 6.0635$ feet.

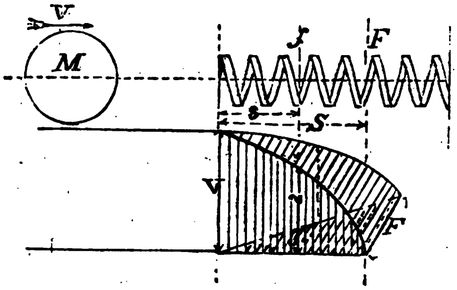
The work in lifting the weight will be

$$K = 50 \times 6.0635 = 303.175 \text{ foot-pounds.}$$

§ 107. ON IRREGULAR WORK IN WHICH BOTH FORCE AND VELOCITY ARE VARIABLE.

Assume the case of a body M moving with a velocity V against a spring, which force of elasticity is F when compressed the space S.

Fig. 131.



It has before been stated that the elastic force of spiral springs is directly as the compression.

The mean force in the space S will then be $\frac{1}{2}F$, or the mean force at any compression s will be $\frac{Fs}{2S}$.

The work done in the compression S will be $K = \frac{FS}{2}$. . 1

The work done in the compression s will be $k = \frac{Fs^2}{2S}$. . . 2

The work consumed on the moving mass is $\frac{MV^2}{2}$. . . 3

When the mass has compressed the spring the space s, its velocity will be reduced to v, and the remaining work required to bring the

v mass to rest will be $\frac{Mv^2}{2}$. 4

Therefore, the work done in compressing the spring the space s will be

$$k = \frac{Fs^2}{2S} = \frac{MV^2}{2} - \frac{Mv^2}{2}. \quad . \quad . \quad . \quad 5$$

$$\frac{Fs^2}{S} = MV^2 - Mv^2, \qquad Mv^2 = MV^2 - \frac{Fs^2}{S}.$$

$$v = \sqrt{V^2 - \frac{Fs^2}{MS}} \quad . \quad . \quad . \quad . \quad 6$$

This formula gives the velocity v with which the mass M compresses the spring at the space s.

The figure under the spring represents the nature of the work performed in compressing the spring, supposing that

$$2K = MV^2 = FS. \quad . \quad . \quad . \quad . \quad 7$$

The vertical lines represent the velocity, and the dotted lines in the base the force of the spring at the corresponding compression s.

The power in operation at any compression s is represented by the section fv of the figure.

The cubic contents of the figure represents the work.

Example 1. It is found by experiment that the spring can be compressed $S = 1$ foot by a force $F = 360$ pounds. Required the velocity of the mass $M = 12$ matts. to compress the spring one foot?

$$MV^2 = FS, \qquad V = \sqrt{\frac{FS}{M}} = \sqrt{\frac{360 \times 1}{12}} = 5.477 \text{ feet per second.}$$

Example 2. Required the velocity v at the compression $s = 0.5$ of the foot?

$$\text{Formula 6.} \quad v = \sqrt{5.477^2 - \frac{360 \times 0.5^2}{12 \times 1}} = 4.75 \text{ feet per second.}$$

The force of the spring at this velocity and compression is

$$f = \frac{Fs}{S} = \frac{360 \times 0.5}{1} = 180 \text{ pounds,}$$

and the power in operation at that moment is

$$P = fv = 4.75 \times 180 = 855 \text{ effects, or } \frac{855}{550} = 1.55 \text{ horse-power.}$$

Example 3. A mass $M = 6$ matts. moving against the spring with a velocity $V = 4$ feet per second. Required its velocity when the spring is compressed $s = 0.6$ feet?

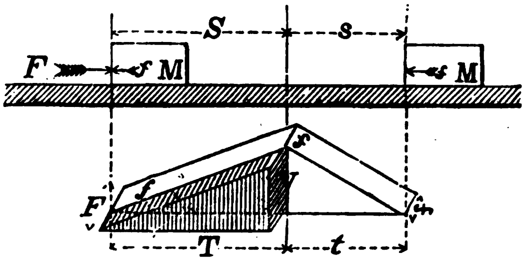
$$\text{Formula 6.} \quad v = \sqrt{4^2 - \frac{360 \times 0.6^2}{6 \times 1}} = -2.37 \text{ feet per second.}$$

The negative sign proves that the mass was not able to compress the spring $s = 0.6$, for which is wanted an additional velocity of 2.37 feet per second to V; or $V = 6.37$ feet per second would just compress the spring that much but no more, and the velocity v would then be 0.

§ 108. THE MASS ACTED UPON BY TWO OPPOSING FORCES.

The force F acts on a mass M sliding on the base B with friction f. The mass is thus acted upon by two opposite forces, F and f, of which

Fig. 132.



F must be greater than f before the mass can be moved, and the force which sets the mass in motion is $F-f$. The force F acts on the mass only in the space S, when the work of the force will be

$$K = F S. \quad . \quad . \quad . \quad . \quad . \quad 1$$

The work done by the friction will be

$$k = f S. \quad . \quad . \quad . \quad . \quad . \quad 2$$

The work utilized in giving the mass M the velocity V in the space S will be

$$K - k = S(F-f) = \frac{MV^2}{2}. \quad . \quad . \quad . \quad 3$$

of which

$$V = \sqrt{\frac{2S(F-f)}{M}}. \quad . \quad . \quad . \quad . \quad 4$$

$$M : (F-f) = T : V. \quad . \quad . \quad . \quad . \quad 5$$

$$T = \frac{MV}{F-f} = \frac{2S}{V}. \quad . \quad . \quad . \quad . \quad 6$$

$$V = \frac{T(F-f)}{M} = \sqrt{\frac{2S(F-f)}{M}}. \quad . \quad . \quad . \quad 7$$

of which $$T = \frac{2SM}{F-f}. \quad \ldots\ldots \quad 8$$

$$S = \frac{T(F-f)}{2M} = \frac{MV^2}{2(F-f)}. \quad \ldots \quad 9$$

$$M = \frac{T(F-f)}{2S} = \frac{(F-f)}{V}\sqrt{T}. \quad \ldots \quad 10$$

$$F = \frac{2\overline{MS}}{T} + f = \frac{MV}{\sqrt{T}} + f. \quad \ldots \quad 11$$

The prism $F\,V\,T$ represents the total work K of the force F and Formula 1, of which the work consumed by the friction is represented by the light part of the prism $f\,V\,T$ and Formula 2.

The shaded part of the prism represents the work utilized in giving the mass M the velocity V, which corresponds with Formula 3.

The second prism $f\,V\,t$ represents the work of friction consumed in bringing the mass to rest after the force F ceased to act. The cubic content of the prism $f\,Vt$ is equal to that of the shaded part of the first prism which set the body in motion.

Example 1. The force $F = 160$ pounds, $M = 64$ matts., and the friction $f = 100$ pounds. What time is required to move the mass M a space $S = 20$ feet? and what will be the velocity V? at the end of that space?

Formula 7. $$V = \sqrt{\frac{2 \times 20(160-100)}{64}} = 6.125 \text{ feet per second,}$$

the required velocity at the end of the space S.

Formula 8. $$T = \frac{2 \times 20 \times 64}{160-100} = 42.66 \text{ seconds,}$$

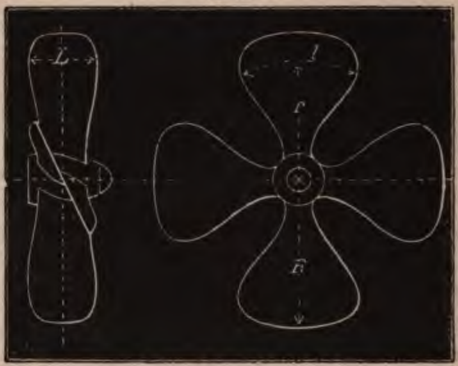
the time required to move the body $S = 20$ feet.

M

§ 109. ON FRICTION OF SCREW PROPELLERS WORKING IN WATER.

The friction of screw propellers running in water is a considerable item of the propelling power, and is well worthy of attention when speed of the vessel and economy of fuel are desired. The solution of the problem furnishes a good example of the value of the calculus in dynamics.

Fig. 133.



Notation of Letters.

P = pitch of the propeller in feet.
R = radius of the propeller in feet.
r = any radius less than R.
l = length of a helix for one whole convolution in feet, at the radius r.
A = area of the helicoidal surface for one whole convolution in square feet.
N = number of blades of the propeller.
n = number of revolutions per minute of the propeller.
v = velocity of the helix l in feet per second.
h = horse-power of friction.
f = friction in pounds.
∂ = differential.

The friction in pounds per square foot of surface of cast-iron and brass, of rough castings, and also of smooth surfaces, filed or ground,

but not polished, is approximated as follows when the surface moves with the velocity of one foot per second:

Friction Surface.	f'
Rough cast-iron	0.0045
Smooth cast-iron	0.0040
Brass, rough casting	8.0040
Brass, smooth surface	0.0030

The friction for rough cast-iron will then be, in pounds,

$$f = 0.0045\, A\, v^2. \quad . \quad . \quad . \quad . \quad . \quad 1$$

The differential area of the helicoidal surface for one convolution will be

$$\partial A = l\, \partial r, \quad . \quad . \quad . \quad . \quad . \quad 2$$

and the differential friction in pounds will be

$$\partial f = 0.0045\, v^2\, l\, \partial r. \quad . \quad . \quad . \quad . \quad . \quad 3$$

This force of friction multiplied by its velocity will be power in effects, and divided by 550 will be horse-power, when the differential horse-power will be

$$\partial h = \frac{0.0045\, v^3\, l\, \partial r}{550}. \quad . \quad . \quad . \quad . \quad . \quad 4$$

The velocity $\quad v = \dfrac{l n}{60} \quad$ and $\quad v^3 = \dfrac{l^3 n^3}{60^3}. \quad . \quad . \quad . \quad 5$

The differential horse-power will then be

$$\partial h = \frac{0.0045\, n^{34}\, l\, \partial r}{550 \times 60^3}. \quad . \quad . \quad . \quad . \quad . \quad 6$$

Call $\quad X = \dfrac{0.0045\, n^3}{550 \times 60^3} = \dfrac{n^3}{26{,}400{,}000{,}000}. \quad . \quad . \quad . \quad 7$

Then $\quad \partial h = X\, l^4\, \partial r, \quad . \quad . \quad . \quad . \quad . \quad 8$

and $\quad h = X \int l^4\, \partial r. \quad . \quad . \quad . \quad . \quad . \quad 9$

But $\quad l = \sqrt{4\, \pi^2\, r^2 + P^2}, \quad$ and $\quad l^4 = (4\, \pi^2\, r^2 + P^2)^2. \quad . \ 10$

Insert Formula 10 in Formula 9, and we have

$$h = X \int (4\, \pi^2\, r^2 + P)^2\, \partial r. \quad . \quad . \quad . \ 11$$

$$(4\, \pi^2\, r^2 + P^2)^2 = 16\, \pi^4\, r^4 + 8\, \pi^2\, r^2\, P^2 + P^4. \quad . \quad . \ 12$$

Then $\quad h = X \int (16\, \pi^4\, r^4 + 8\, \pi^2\, r^2\, P^2 + P^4)\, \partial r. \quad . \quad . \ 13$

By integrating each term in the parenthesis we have

$$h = X\left(\frac{16\,\pi^4\,r^5}{5} + \frac{8\,\pi^2\,r^3\,P^2}{3} + r\,P^4\right) + C. \quad . \quad . \quad 14$$

$$\frac{16\,\pi^4\,r^5}{5} = 311.71\,r^5, \text{ and } \frac{8\,\pi^2\,r^3\,P^2}{3} = 26.319\,r^3\,P^4. \quad . \quad 15$$

Integrate the friction horse-power from the centre of the propeller to the periphery of radius R, then when $r = 0$, $C = 0$.

Insert the Formula 7 for X in Formula 14, with the values 15, and we have the frictional horse-power for one convolution of the screw and for one side of the helicoidal surface,

$$h = \frac{n^3\,R}{26,400,000,000}(311.71\,R^4 + 26.319\,R^2\,P^2 + P^4). \quad . \quad 16$$

The helicoidal surface of screw propellers is generally cut up into small portions by a number of blades, each of a fraction of the pitch, and when the helicoidal surface is counted on both sides of the blade, the friction horse-power will be

$$h = \frac{R\,L\,N\,n^3}{13,200,000,000\,P}(311.71\,R^4 + 26.319\,R^2\,P^2 + P^4) \quad . \quad 17$$

This formula includes both the dragging and rotary friction horse-power of a propeller of rough cast-iron.

Call f' = friction in pounds per square foot of surface moving with a velocity of one foot per second, and the friction horse-power will be

$$h = \frac{f'\,R\,L\,N\,n^3}{59,400,000\,P}(311.71\,R^4 + 26.319\,R^2\,P^2 + P^4). \quad . \quad 18$$

Example. Required the friction horse-power of a propeller of the following dimensions:

Diameter of propeller, 20 feet, or............$R =$ 6 feet.
Pitch..$P =$ 18 feet.
Length in the direction of axis...............$L = 2.4$ feet.
Number of blades...................................$N =$ 4.
Revolutions per minute.........................$n = 60$.

The horse-power consumed by friction will then be, for rough cast-iron surface,

$$h = \frac{0.0045 \times 6 \times 2.4 \times 4 \times 60^3}{59,400,000 \times 18}(311.71 \times 6^4 + 26.319 \times 6^2 \times 18^2 + 18^4) = 10.8$$

horse-power of friction.

In the year 1850 an experimental steamer was built in Kensington, Philadelphia, which was expected to make 20 to 30 miles per hour. The propeller was about 4 feet in diameter, with only one blade, extending the whole convolution of the circle, and with a very fine pitch of about 6 inches. (The author does not remember the exact dimensions.) The propeller was expected to make 500 revolutions per minute.

Example.—Required the friction horse-power of the above-described propeller. $R=2$ feet, $P=0.5$ feet, $n=500$, $L=0.5$, $N=1$. Surface, rough cast-iron.

$$h=\frac{0.0045\times 2\times 0.5\times 1\times 500^3}{59{,}400{,}000\times 0.5}(311.71\times 2^4+26.319\times 2^2\times 0.5^2+0.5^4)=76$$

horse-power, nearly.

The power of the engine counted from the size of the steam boiler, did not amount to more than 50 horse-power, and the result was that when the trial trip came off the steamer could hardly crawl up against the tide.

The building of the steamer was kept in the greatest secresy, and her performance was expected to astonish the world.

There was another experimental steamer built in Kensington in the year 1864, in which several curious propellers were placed on each side of the vessel, which also turned out a failure on account of the friction of the propellers in the water being too great.

A fine-pitched propeller has more friction-power than one with sharp pitch for equal speed of vessel.

The proper pitch for propelling steamboats should be between two and two and a half times the diameter of the propeller. The sharpest vessel should have the sharpest pitch of propeller.

For the proper proportions and construction of screw-propellers, see "Nystrom's Pocket-Book of Mechanics," thirteenth edition.

§ 110. GYRATION.

Gyration means Circular Motion.

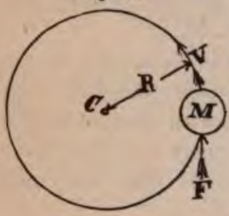
The term is used in dynamics of matter in circular motion to designate the mean effect of all the particles in a revolving body or system of bodies.

In motion of translation all the particles of a body move with equal velocity in straight and parallel lines, but in circular motion the particles move with different velocities in different circles, of which one is called the *circle of gyration*.

§ 111. DYNAMICS OF MATTER IN CIRCULAR MOTION.

Fig. 134 represents a mass M free to move around the centre C in the circle of radius R. A constant force F is applied in the direction of the tangent of the circle to move the mass, the dynamics of which will be the same as if the mass moved in a straight line or

Fig. 134.



$$F : M = V : T, \qquad \text{and } V = \frac{F\,T}{M}.$$

The velocity V of the mass M will be in the direction of the circle, which periphery is $2\,\pi\,R$.

Circular velocity is generally expressed in revolutions per minute, and denoted by the letter n. When the radius R is expressed in feet the circular velocity will be

$$V = \frac{2\,\pi\,R\,n}{60}, \qquad \text{and } n = \frac{60\ V}{2\,\pi\,R}.$$

Example 1. A force $F = 36$ pounds is applied on the mass $M = 48$ matts. for a time $T = 9$ seconds. Required the velocity V? and revolutions per minute? with which the mass will continue to rotate after the force ceased to act? The radius of the circle being $R = 2$ feet.

$$V = \frac{36 \times 9}{48} = 6.75 \text{ feet per second.}$$

$$n = \frac{60 \times 6.75}{2 \times 3.14 \times 2} = 32.6 \text{ revolutions per minute.}$$

The mass M will continue to rotate with this velocity until some force F' is applied in opposite direction to retard the motion and finally bring the mass to rest.

Example 2. What force F' is required to stop the rotation of the mass M in a time of $T=4$ seconds?

$$F'=\frac{M\,V}{T}=\frac{48\times 6.75}{4}=80 \text{ pounds.}$$

The primitive work consumed in setting the body in rotation is equal to the realized work by which the rotation is stopped, like in straight linear motion, the

$$\text{Work} \quad K=\frac{F\,V\,T}{2}=\frac{M\,V^2}{2}=\frac{M(2\,\pi\,R\,n)^2}{2\times 60^2}.$$

$$K=\frac{M\,R^2\,n^2}{182.377}=0.00548314\ M\,R^2\,n^2, \text{ the work of rotation.}$$

This formula gives the primitive or realized work of a rotating mass, as illustrated by Fig. 134.

When the mass is expressed by weight W, the formula will be

$$\text{Work} \quad K=\frac{W\,R^2\,n^2}{32.17\times 182.377}=\frac{W\,R^2\,n^2}{5867.16}.$$

When the force F is applied on a radius r, and the centre of the mass M rotates with a radius R, as represented by Fig. 136, the formulas will be as follows:

$F=\dfrac{M\,V\,R}{r\,T}$	1	$F=\dfrac{M\,2\,\pi\,R^2\,n}{60\,r\,T}$	5
$M=\dfrac{F\,r\,T}{R\,V}$	2	$M=\dfrac{60\,F\,r\,T}{2\,\pi\,R\,n}$	6
$V=\dfrac{F\,r\,T}{M\,R}$	3	$n=\dfrac{60\,F\,r\,T}{2\,\pi\,R^2 M}$	7
$T=\dfrac{M\,V}{F\,r}$	4	$T=\dfrac{M\,2\,\pi\,R^2\,n}{60\,F\,r}$	8

Example 1. A mass $M=96$ matts. is to be put into a circular motion of $n=360$ revolutions per minute around a radius $R=3$ feet. The force F which sets the mass in rotation acts on a lever or crank of radius $r=0.5$ of a foot. What force is required to give the mass that circular velocity in a time $T=10$ seconds?

$$\text{Formula ?} \quad F=\frac{96\times 2\times 3.14\times 3^2\times 360}{60\times 0.5\times 10}=6511.1 \text{ pounds.}$$

§ 112. CENTRE OF GYRATION.

Centre of gyration in a revolving body is a point in which, if all the matter were there contained, it would have the same dynamic effect as when distributed around that centre.

The centre of gyration describes the circle of gyration.

The centre of gyration is always outside of the centre of gravity of the revolving body.

The less space the body occupies in the circle of gyration, and the greater the radius of revolution is, the nearer does the centre of gyration approach the centre of gravity of the body.

§ 113. RADIUS OF GYRATION.

The radius of gyration is the distance from the centre of rotation to the centre of gyration in a revolving body.

The radius of gyration will herein be denoted by the letter X, to distinguish it from other radii.

§ 114. MOMENT OF INERTIA.

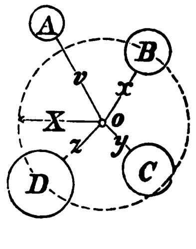
Moment of inertia is either weight, mass, volume or surface, multiplied by the square of its radius of gyration.

The moment of inertia will hereafter be denoted by the letter E, and may be expressed in either of the following units:

$E = W X^2$ in square foot-pounds, or sq. ft. lbs.
$E = M X^2$ in square foot-matts., or sq. ft. mts.
$E = Q X^2$ in square foot-volumes, or sq. ft. vol.
$E = O X^2$ in square foot-square, or sq. ft. sq.

Moment of inertia is used for finding the radius of gyration.

Fig. 135.



Let Q denote the volume of any system of bodies, and X its radius of gyration rotating around the centre o, and A, B, C, D, etc. represent the several volumes of respective radii of gyration v, x, y, z, etc.

Then $Q X^2 = A v^2 + B x^2 + C y^2 + D z^2$, etc.

The radius of gyration of the system will then be

$$X = \sqrt{\frac{A v^2 + B x^2 + C y^2 + D z^2, \text{ etc.}}{Q}}.$$

We have learned that the work of a revolving body is

$$K = \frac{M R^2 n^2}{182.377}, \quad \text{or} \quad K = \frac{W R^2 n^2}{5867.16},$$

in which R means the radius of gyration, $M R^2$ and $W R^2$ means the moment of inertia.

When the moment of inertia E is given the work in the revolving body will be

$$K = \frac{E n^2}{182.377}, \quad \text{or} \quad K = \frac{E n^2}{5867.16}.$$

The moment of inertia is a constant quantity in a body or system of bodies rigid to an axis of rotation.

Moment of inertia is $E = M X^2$. 1

Square radius of gyration is $X^2 = \frac{M}{E}$. 2

Radius of gyration is then $X = \sqrt{\frac{M}{E}}$. 3

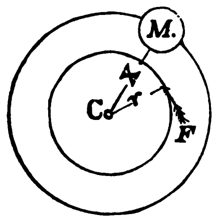
Mass of the body will be $M = \frac{E}{X^2}$. 4

§ 116. FORCE OF INERTIA.

The force of inertia is equal to any force applied to change the motion of a body.

Let a constant force F be applied on a lever r to change the rotary motion of the mass M reacting with its force of inertia I on the lever or radius of gyration X.

Fig. 136.



Then, $F : I = X : r$,

and the static momentums $F r = I X$.

The force of inertia will be $I = \frac{F r}{X}$.

Let T denote the time in seconds, in which the velocity of the mass M is changed from V to v feet per second, or from N to n revolutions per minute.

Then, $I : M = (V - v) : T$,

from which the force of inertia will be

$$I = \frac{M(V-v)}{T} = \frac{Fr}{X}.$$

Static momentum $\quad Fr = IX = \dfrac{MX(V-v)}{T}.$

$$V = \frac{2\pi XN}{60}, \quad v = \frac{2\pi Xn}{60}, \quad \text{and} \quad (V-v) = \frac{2\pi X}{60}(N-n).$$

Static momentum $\quad Fr = IX = \dfrac{2\pi MX^2}{60\,T}(N-n).$

The time T in which the angular velocity is changed from V to v, or revolutions from N to n, will be

$$T = \frac{MX(V-v)}{Fr}, \quad \text{or} \quad T = \frac{2\pi MX^2(N-n)}{60\,Fr}.$$

The change of angular velocity in the time T will be

$$(V-v) = \frac{FrT}{MX}, \quad \text{or} \quad (N-n) = \frac{60\,FrT}{2\pi MX^2}.$$

The force F required on the crank will be

$$F = \frac{MX(V-v)}{Tr}, \quad \text{or} \quad F = \frac{2\pi MX^2(N-n)}{60\,Tr}.$$

Example. A body weighing 2255 pounds, or mass $M = 70$ matts., revolving with a radius of gyration $X = 5$ feet, and making $N = 120$ revolutions per minute, is acted upon by a force $F = 96$ pounds on a lever $r = 0.75$ of a foot. What time T is required to reduce the revolution to $n = 60$ per minute?

$$T = \frac{70 \times 2 \times 3.14 \times 5^2}{60 \times 96 \times 0.75}(120-60) = 152.6 \text{ seconds},$$

or 2 minutes 32.6 seconds, the answer.

Example. A mass $M = 360$ matts. is revolving with a velocity of $V = 30$ feet per second in a circle of $X = 12$ feet radius of gyration, when a force $F = 90$ pounds acting on a crank $r = 1.25$ feet is applied

in $T=180$ seconds to increase the angular velocity. Required the increased velocity V?

$$(V-v)=\frac{90\times1.25\times180}{360\times12}=4.1.$$

$$V-v=4.1, \quad \text{or} \quad V=4.1+30=34.1 \text{ feet per second,}$$

the velocity required.

The force of inertia of any revolving body in the circle of gyration will be

$$I=\frac{M\,2\,\pi\,X}{60\,T}(N-n).$$

Example. A fly-wheel weighing 5400 pounds, or $M=167.84$ matts., is making $N=120$ revolutions per minute, with a radius of gyration $X=4.5$ feet. The angular velocity of the wheel is to be reduced to $n=15$ revolutions per minute in a time $T=90$ seconds. What force I must be applied in the tangent of the circle of gyration to reduce the revolutions from 120 to 15 in the time 90 seconds?

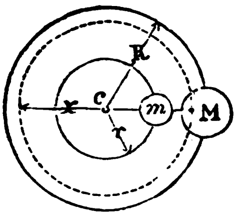
$$I=\frac{167.84\times2\times3.14\times4.5}{60\times90}(120-15)=92 \text{ pounds.}$$

This is the force of inertia under the conditions given in the example.

§ 116. RADIUS OF GYRATION AND MOMENT OF INERTIA.

Let two masses M and m rotate around a common centre C in circles of gyration of radii R and r, and the two masses being rigid so as to rotate with a common angular velocity. Then the moment of inertia of each mass will be $M\,R^2$ and $m\,r^2$, and of the two masses

Fig. 137.



$$E=(M+m)X^2=M\,R^2+m\,r^2.$$

From this formula we have the radius of gyration to be

$$X=\sqrt{\frac{M\,R^2+m\,r^2}{M+m}}.$$

Let Z denote the distance of the centre of gravity of the two masses from the centre C, and we have the static momentum

$$Z(M+m)=M\,R+m\,r, \qquad \text{of which } Z=\frac{M\,R+m\,r}{M+m}.$$

It is to be proved that the centre of gyration is outside of the centre of gravity of the two masses, or that $X > Z$.

$$X^2 = \frac{M R^2 + m r^2}{M+m}. \qquad Z^2 = \left(\frac{M R + m r}{M+m}\right)^2.$$

$$\frac{M R^2 + m r^2}{M+m} > \frac{(M R + mr)^2}{(M+m)(M+m)}.$$

Reject the common denominator $M+m$.

$$M R^2 + m r^2 > \frac{(M R + m r)^2}{M+m}.$$

Multiply both members by $M+m$.

$$(M+m)(M R^2 + m r^2) > (M R + m r)^2.$$

$$M^2 R^2 + M m R^2 + M m r^2 + m^2 r^2 > M^2 R^2 + 2 M R m r + m^2 r^2.$$

$$M m R^2 + M m r^2 > 2 M R m r.$$

$R^2 + r^2 > 2 R r$; which is true, because if $R > r$, the two squares $R^2 + r^2$ must be greater than two rectangles $R r$.

§ 117. GYRATION OF IRREGULAR BODIES.

To find the radius of gyration and moment of inertia of an irregular body revolving around the centre C, Fig. 138.

Draw concentric circles at equal distances r apart to represent cylindrical sections of the body concentric with the axis C.

Fig. 138.

The first and last division should be only $\frac{1}{2} r$. The more divisions made the more correctly will the radius of gyration and moment of inertia be ascertained.

The area of each cylindrical section a, b, c and d will then represent the mass of rotation at the respective section.

Let O denote the sum of all the areas of the sections, and X = radius of gyration of the body. The sum of the moments of inertia of the sections will be equal to $O X^2$.

$$O = a + b + c + d.$$

Moments of inertia, $O X^2 = a r^2 + b(2 r)^2 + c(3 r)^2 + d(4 r)^2$.

$$X^2 = \frac{r^2}{O}(a + 4b + 9c + 16d).$$

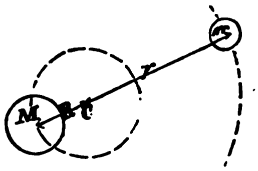
Rad. gyration, $X = r\sqrt{\frac{1}{O}(a + 4b + 9c + 16d)}$.

§ 118. EQUILIBRIUM AND DISTURBANCE OF GYRATION.

A body or system of bodies revolving around an axis c upon which it may be perfectly balanced in regard to gravity, but when rotating with a variable velocity, the equilibrium is disturbed in the axis of rotation.

Two masses M and m revolving around their common centre of gravity o, will be in gyratic equilibrium only when their angular velocity is uniform, in which case the equilibrium is not disturbed in the axis of rotation; but when the angular velocity is irregular, there is a tendency to change the axis of rotation, and the equilibrium of gyration is disturbed.

Fig. 139.



Let R and r denote the respective radii of gyration of the masses M and m, and X = radius of gyration of the system.

Suppose the bodies to be balanced so that $M:m=r:R$, or $MR=mr$, in which case their dynamic momentum will be alike, or $MV=mv$; but the work in each body will not be alike.

From § 112 we have the works in the revolving bodies to be

$$K=\frac{MR^2n^2}{182.377}, \text{ and } k=\frac{mr^2n^2}{182.377}.$$

Insert MR in the work k, and mr in the work K, which will be

$$K=\frac{mrRn^2}{182.377}, \text{ and } k=\frac{MRrn^2}{182.377}.$$

$$n^2=\frac{182.377\,K}{mrR}=\frac{182.377\,k}{MRr}.$$

$$KMRr=kmrR, \text{ or } K:k=m:M.$$

That is to say, the work in the bodies are inversely as the masses; therefore, the work which sets the unequal masses into rotation is unequally divided in the centre of rotation, which causes disturbance.

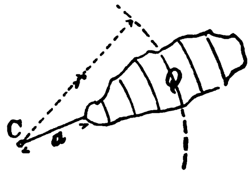
When the masses rotate with a uniform velocity, there is no work transmitted to or from them through the centre of rotation, and there can consequently be no disturbance.

Any change in the angular velocity will cause a disturbance in the journals, which disturbance cannot be avoided except by making $MR^2=mr^2$; but, then the rotating system will cause a disturbance during uniform velocity, because it is then not balanced for gravity.

§ 119. GYRATION TREATED BY THE CALCULUS.

Any body or system of bodies may be considered to be composed of an infinite number of cylindrical sections concentric with the common axis C of rotation.

Fig. 140.



Let O denote the area of any such section rotating around the axis o, with the radius r.
Q = cubic content of the body.

Then $$\delta Q = O\,\delta r.$$

Moment inertia $$Q X^2 = \int O\,r^2\,\delta r + C.$$

Radius gyration $$X = \sqrt{\int \frac{O\,r^2\,\delta r}{Q} + C}.$$

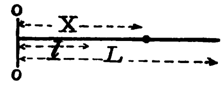
In order to bring the formulas into a practical shape, it is necessary to know the variation of the section O in relation to r.

§ 120. GYRATION OF A STRAIGHT LINE OR ROD.

Centre of Gyration and Moment of Inertia.

Let a straight line or thin parallel rod L be attached with one end, and at right angle to the axis of rotation $o\ o$. As the rod is parallel, its section O at any distance l is a constant quantity.

Fig. 141.



$$\delta Q\,X^2 = O\,l^2\,\delta l.$$

Moment inertia $$Q X^2 = \int^{l=L} O\,l^2\,\delta l = \tfrac{1}{3} O\,L^3 \text{ sq. ft. vol.}$$

$$Q = O\,L, \text{ and } O = \frac{Q}{L}.$$

Moment inertia $$Q X^2 = \tfrac{1}{3} Q\,L^2.$$

Radius gyration $$X = \sqrt{\frac{Q\,L^2}{3\,Q}} = \sqrt{\frac{L^2}{3}} = L\sqrt{\tfrac{1}{3}}.$$

Radius gyration $$X = 0.5775\ L.$$

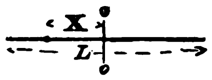
Example. A line or parallel rod is $L = 12$ feet long, and revolves around one of its ends. Required the radius of gyration?

$$X = 0.5775 \times 12 = 6.93 \text{ feet.}$$

§ 121. GYRATION OF A BALANCED LINE OR PARALLEL ROD.

The radius of gyration of a line or parallel rod revolving around an axis passing through the centre of gravity at right angle to the line will be

Fig. 142.

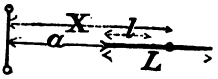


$$X = \tfrac{1}{2}\times 0.5775,\quad L = 0.28875\ L.$$

§ 122. GYRATION OF A STRAIGHT LINE NOT EXTENDING TO THE AXIS OF ROTATION.

When the revolving line or parallel rod does not extend to the axis of rotation, the radius of gyration is found as follows:

Fig. 143.



The radius of gyration of each elementary section of the line or rod, will be $(a+l)^2\,\partial l$.

$$\text{Moment inertia}\quad L\,X^2 = \int (a+l)^2\,\partial l.$$

$$(a+l)^2 = a^2 + 2\,a\,l + l^2.$$

$$L\,X^2 = \int_a^{a+l} (a^2 + 2\,a\,l + l^2)\,\partial l = a^2\,L + a\,L^2 + \tfrac{1}{3}L^3 = L(a^2 + a\,L + \tfrac{1}{3}L^2).$$

$$\text{Radius gyration}\quad X = \sqrt{a^2 + a\,L + \tfrac{1}{3}L^2}.$$

§ 123. CENTRE OF GYRATION REFERRED TO CENTRE OF GRAVITY.

Having given the radius of gyration of a body or system of bodies when the axis of rotation passes through the centre of gravity, to find the radius of gyration of the same when the centre of gravity rotates with a radius R around an axis parallel with the former axis.

Referring to the preceding figure, the radius R of the centre of gravity of the rod will be $R = a + \frac{1}{2}L$.

$$a = R - \tfrac{1}{2}L,\quad\text{and}\quad a^2 = R^2 - R\,L + \tfrac{1}{4}L^2.$$

The radius of gyration of the preceding figure is

$$X = \sqrt{a^2 + a\,L + \tfrac{1}{3}L^2}.$$

Insert the above values of a and a^2, and we have

$$X = \sqrt{R^2 - L\,R + \tfrac{1}{4}L^2 + L(R - \tfrac{1}{2}L) + \tfrac{1}{3}L^2}.$$

$$\text{Radius gyration}\quad X = \sqrt{R^2 + \tfrac{1}{12}L^2}.$$

This formula gives the centre of gyration for any parallel figure of length L.

$$\sqrt{\tfrac{1}{12}\,L^2} = 0.28875\ L,$$

which is the radius of gyration of a line L rotating around its centre of gravity. Therefore, if r = radius of gyration of a body or system of bodies rotating around its centre of gravity, and R = radius of centre of gravity of the same system rotating around another axis parallel to the former, then

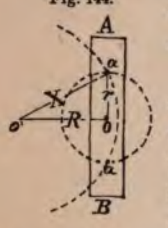
$$\text{Radius gyration} \quad X = \sqrt{R^2 + r^2}.$$

This formula will hold good for any form of body or system of bodies.

§ 124. GYRATION OF A BODY REVOLVING OUTSIDE OF ITS AXIS OF ROTATION.

The illustration represents the plan of the circles of rotation. The body $A\ B$ is first supposed to revolve around its centre of gravity o when its centre of gyration a describes the dotted small circle of radius r.

Fig. 144.



Now let the body revolve around the centre o', with a radius R from the centre of gravity o, so that the radii R and r are at right angles to one another. Then the radius of gyration will be $X = o'\ a$, which is the hypotenuse of the catheters R and r, and the centre of gyration will in this case be at b, describing the dotted large circle.

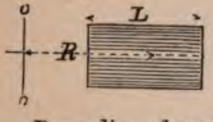
This is an illustration of the formula in the preceding paragraph,

$$X = \sqrt{R^2 + r^2}.$$

The body $A\ B$ may revolve in any position in regard to the circles of gyration, with the condition that the two axes o and o' must be parallel.

§ 125. GYRATION OF A RECTANGULAR PLANE.

Fig. 145.



The radius of gyration of any number of parallel lines rigid into a rectangular plane, which ends are parallel with the axis of rotation, is the same as that for a single line.

R = radius of centre of gravity of the plane.

r = radius of gyration of the plane when the axis of rotation passes through the centre of gravity and is parallel to the axis o, o.

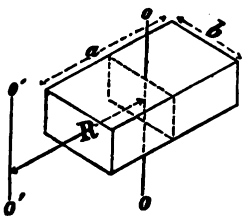
O = area of the rectangular plane.

$$\text{Moment inertia} \quad E = O(R^2 + r^2) = O(R^2 + \tfrac{1}{12} L^2).$$

$$\text{Radius gyration} \quad X = \sqrt{R^2 + r^2} = \sqrt{R^2 + \tfrac{1}{12} L^2}.$$

§ 126. GYRATION OF A PARALLELOPIPEDON.

Fig. 146.



To find the radius of gyration and moment of inertia of a parallelopipedon when the axis o o, of rotation passes through the centre of gravity and at right angles to either one of its sides.

The radius of gyration of the dotted section passing through the axis and centre of gravity is (§ 121)

$$0.28875\, b.$$

The radius gyration of any other parallel section at a distance l from the axis is

$$\sqrt{l^2+(0.28875\, b)^2}$$

$a\, b$ = area of one side of the parallelopipedon in the plane of rotation. Then the differential moment inertia will be

$$\delta(a\, b)X^2 = [l^2+(0.28875\, b)^2]b\, \delta l.$$

$$\text{Moment inertia}\quad a\, b\, X^2 = \int [l^2+(0.28875\, b)^2]b\, \delta l.$$

$$a\, b\, X^2 = (\tfrac{1}{3}\, l^3 + \tfrac{1}{12}\, b^2\, l)b.$$

Integrate the moment inertia from $l = o$ to $l = \frac{1}{2}\, a$, and we have

$$a\, b\, x^2 = (\tfrac{1}{3} \times \tfrac{1}{8}\, a^3 + \tfrac{1}{12}\, b^2\, \tfrac{1}{2}\, a)b.$$

$$X^2 = \left(\frac{a^2}{24} + \frac{b^2}{24}\right).$$

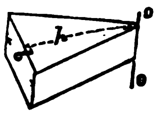
$$\text{Radius gyration}\quad X = \sqrt{\frac{a^2+b^2}{24}}.$$

Let the same parallelopipedon revolve around an axis o', o' parallel with o, o, and R = radius of the centre of gravity. Then the radius of gyration will be (§ 123 and 124)

$$X = \sqrt{R^2 + \frac{a^2+b^2}{24}}.$$

§ 127. GYRATION OF A LATERAL TRIANGLE OR SOLID WEDGE ROTATING AROUND ITS VERTEX.

Fig. 147.



h = height and b = linear base of the triangle.
a = any breadth parallel with b, but at a distance l from o, o.
$0 = \frac{1}{2}$ breadth, the area of the triangle.

The differential moment of inertia will then be

$$\partial\, O\, X^2 = a\, l^2\, \partial\, l.$$

$$a : b = l : h, \quad \text{of which} \quad a = \frac{b\, l}{h}.$$

$$\partial O\, X^2 = \frac{b\, l^3}{h} \partial l.$$

Moment inertia, $O\, X^2 = \int \frac{b\, l^3}{h} \partial\, l = \frac{b\, l^4}{4\, h} = \frac{b\, h^3}{4}.$

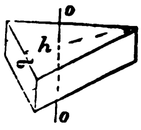
$$\tfrac{1}{2}\, b\, h\, X^2 = \frac{b\, h^3}{4}.$$

$$X^2 = \frac{h^2}{2}.$$

Rad. gyration, $X^2 = h\sqrt{\tfrac{1}{2}} = 0.70107\ h.$

§ 128. GYRATION OF A LATERAL TRIANGLE OR SOLID WEDGE ROTATING AROUND AN AXIS PASSING THROUGH THE CENTRE OF GRAVITY.

Fig. 148.



From the preceding section we know that when the body revolves with its vertex in the axis $o'\,o'$, the rad. gyration is $0.70107h$.

We also know from § 123 that when r = rad. gyration of a body when rotating around its centre of gravity,

$$X = \sqrt{R^2 + r^2}, \qquad \text{of which } r = \sqrt{X^2 - R^2},$$

but $X^2 = \frac{1}{2}\, h^2$, and $R^2 = (\frac{2}{3}\, h)^2 = \frac{4}{9}\, h^2$.

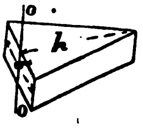
$$r = \sqrt{\tfrac{1}{2}\, h^2 - \tfrac{4}{9}\, h^2} = h\sqrt{\tfrac{1}{2} - \tfrac{4}{9}} = 0.2357h,$$

which is the radius gyration of a triangle or wedge rotating around an axis passing through the centre of gravity.

§ 129. GYRATION OF A WEDGE ROTATING AROUND AN AXIS PASSING THROUGH THE MIDDLE OF ITS BASE.

$$X = \sqrt{R^2 + r^2}.$$

Fig. 149.



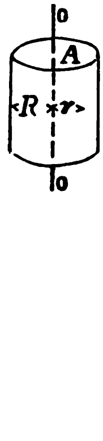
In this case $R = \frac{1}{3} h$, and $R^2 = \frac{1}{9} h^2$. From the preceding paragraph we have $r = 0.2357h$ and $r^2 = \frac{1}{18} h^2$.

$$x = \sqrt{\tfrac{1}{9} h^2 + \tfrac{1}{18} h^2} = h\sqrt{\tfrac{1}{6}} = 0.4082h.$$

§ 130. CIRCULAR PLANE OR SOLID CYLINDER.

Fig. 150 represents a circular plane or solid cylinder rotating around its centre.

Fig. 150.



Let A represent the area of the circular end of the cylinder, and r = any radius less than R. The circumference of the radius r is $2\pi r$, and the differential area

$$\delta A = 2 \pi r \, \delta r.$$

Differential mom. in., $\delta A \; X^2 = 2 \pi r^3 \delta r$.

$$X^2 = \int_{r=0}^{r=R} \frac{2 \pi r^3 \delta r}{A} = \frac{2 \pi R^4}{4 A}.$$

but $\quad A = \pi R^2, \quad$ then $X^2 = \dfrac{2 \pi R^4}{4 \pi R^2} = \dfrac{R^2}{2}$.

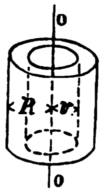
Radius gyration, $\quad X = \sqrt{\dfrac{R^2}{2}} = \dfrac{R}{1.414} = 0.7071 \, R.$

Radius of gyration, $X = 0.7071 \, R$.

§ 131. ANNULAR RING.

This example is the same as the foregoing, except that the formula is integrated from r to R instead of from o to R.

Fig. 151.



$$X^2 = \int_r^R \frac{2 \pi r^3 \delta r}{A} = \frac{2 \pi R^4}{4 \pi R^2} + \frac{2 \pi r^4}{4 \pi r^2}.$$

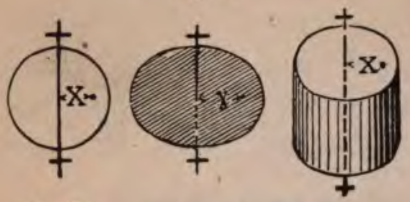
$$X^2 = \frac{R^2}{2} + \frac{r}{2}.$$

Rad. of gyration, $X = \sqrt{\dfrac{R^2 + r^2}{2}}$. 7

The radius of gyration of a circle or a cylindrical surface rotating around its centre or axis is equal to the radius of the circle or cylinder.

§ 132. RADIUS OF GYRATION OF DIFFERENT FIGURES.

Fig. 152.



A circumference rotating around its diameter, and a circular plane or cylinder around its centre. r = radius of the circle.

Radius gyration $X = 0.7071\ r$.

Fig. 153.

A Circular Plane Revolving around its Diameter.

Radius gyration $X = 0.5\ r$.

r = radius of the circle.

Fig. 154.

A Sphere Revolving around its Diameter.

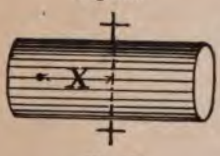
Spherical surface $X = 0.8165\ r$.

A solid sphere $X = 0.6324\ r$.

Fig. 155.

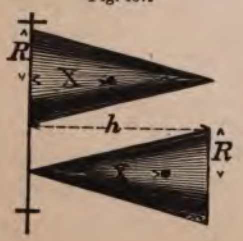
A Cylinder Rotating around one of its Ends.

Radius gyration $X = \sqrt{\dfrac{4\,l^2 + 3\,r^2}{12}}$.

Fig. 156.

A Cylinder Rotating around its Middle.

Radius gyration $X = \sqrt{\dfrac{l^2 + 3\,r^2}{12}}$.

Fig. 157.

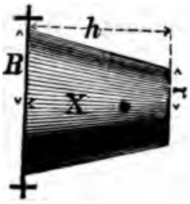


A Cone Revolving around its

Base $X = \sqrt{\dfrac{2\,h^2 + 3\,R^2}{20}}$.

Vertis $X = \sqrt{\dfrac{12\,h^2 + 3\,R^2}{20}}$.

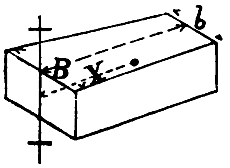
Fig. 158.



A Cone Frustum Rotating around its Base.

$$X = \sqrt{\frac{h}{10}\left(\frac{R^2+3Rr+2.25r^2}{R^2+Rr+r^2}\right)+\frac{3}{20}\left(\frac{R^5-r^5}{R^3-r^3}\right)}.$$

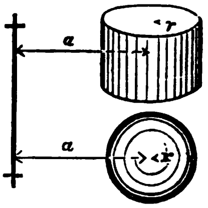
Fig. 159.



A Wedge, or an Arm of a Wheel.

$$X = 0.204\sqrt{12\,l^2+B^2+b^2}.$$

Fig. 160.



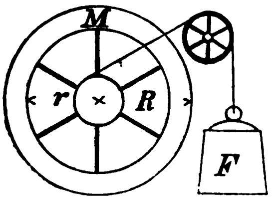
A Cylinder whose Centre Line is parallel with the axis of rotation.

$$X = \sqrt{a^2+\tfrac{1}{2}r^2}.$$

A Sphere Rotating with its Centre a distance a from the axis.

$$X = \sqrt{a^2+0.4\,r^2}.$$

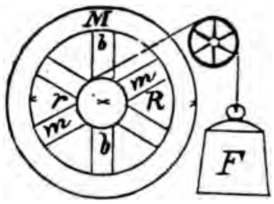
Fig. 161.



A Ring of Square Section, or a fly-wheel of very light arms.

$$X = \sqrt{\frac{R^2+r^2}{2}}.$$

Fig. 162.



To find the Common Radius of gyration of a fly-wheel with arms of considerable weight.

W = weight of the ring of square section.

w = weight of all the arms.

b = breadth of the arms in the direction of rotation in feet.

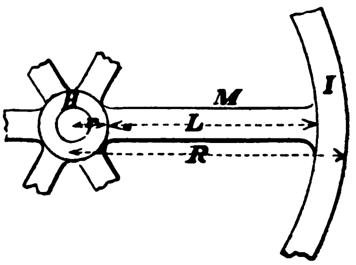
$$X^2(W+v) = W\left(\frac{R^2+r^2}{2}\right)+w\left(\frac{4\,r^2+b^2}{12}\right).$$

$$X=\sqrt{\frac{6\,W(R^2+r^2)+w(4\,r^2+b^2)}{12(\,W+w)}}.$$

§ 133. PARALLEL ARMS OF FLY-WHEELS.

Fig. 163 represents a part of a fly-wheel with hub H, arm M, and ring I.

Fig. 163.



It is required to find the radius of gyration of the arms.

L = length of the parallel arm.
A = area of cross section of the arm.
r = radius of the hub.
l = distance from the centre to any section A.
Q = cubic content of the arm.
y = radius of gyration of the arm.

The momentum of radius of gyration of the consolidated elementary sections A of the arm will be

$$\partial Q\,y^2 = A(r+l)^2\,\partial l.$$

Q is the variable, and not y.

$$y^2=\frac{A}{Q}\int(r+l)^2\,\partial l.$$

$$(r+l)^2=r^2+2\,r\,l+l^2.$$

$$\int_r^{r+L}(r^2+2\,r\,l+l^2)\partial l=r^2\,L+r\,L^2+\tfrac{1}{3}\,L^3=L(r^2+r\,L+\tfrac{1}{3}L^2).$$

$$y^2=\frac{A\,L}{Q}(r^2+r\,L+\tfrac{1}{3}\,L^2),$$

but $Q=A\,L$, and

Radius gyration $\quad y=\sqrt{r^2+r\,L+\tfrac{1}{3}\,L^2}.$

§ 134. RADIUS OF GYRATION OF THE WHOLE FLY-WHEEL.

x = Radius of gyration of the hub, to be calculated from § 130.
O = weight in pounds of the hub.
y = Radius gyration of the arms, to be calculated from § 133.
P = weight in pounds of all the arms.
z = Radius gyration of the outer ring, to be calculated from § 130.
Q = weight in pounds of the ring.
X = Radius gyration of the whole wheel.
W = weight in pounds of the whole wheel.

$$W = O + P + Q.$$

$$WX^2 = Ox^2 + Py^2 + Qz^2.$$

$$X = \sqrt{\frac{Ox^2 + Py^2 + Qz^2}{W}}.$$

Example. Required the radius of a cast-iron fly-wheel of the following dimensions?

Hub, $r = 6$ inches, radius shaft in the hub.
$R = 15$ inches, outside radius of the hub.
$O = 2798$ pounds, weight of the hub.
Arms, $L = 10.75$ feet, length of the arms.
$P = 10650$ pounds, weight of six arms.
Ring, $R = 13$ feet, outer radius of the ring.
$Q = 36500$ pounds.
Wheel, $W = 49948$ pounds, the weight of the whole wheel.

Radius gyration of the hub will be,

$$x = \sqrt{\frac{1.25^2 + 0.5^2}{2}} = 0.952 \text{ of a foot.}$$

Radius gyration of the arms will be,

$$y = \sqrt{1.25^2 + 1.25 \times 10.75 + \tfrac{1}{3} \times 10.75^2} = 7.32 \text{ feet.}$$

Radius gyration of the ring,

$$z = \sqrt{\frac{13^2 + 12^2}{2}} = 12.51 \text{ feet.}$$

Radius gyration of the whole system of the fly-wheel will then be,

$$X=\sqrt{\frac{2798\times0.952^2+10650\times7.32^2+36500\times12.51^2}{49948}}=11.217 \text{ feet,}$$

the radius gyration of the fly-wheel.

This is $12-11.217=0.783$ feet, or 9 inches less radius of gyration than the inner radius of the ring, which latter is generally taken in practice. In this case the radius gyration is only 0.934 of the inner radius, or 0.86 of the outer radius.

§ 135. FLY-WHEELS.

In fly-wheels of ordinary proportions the radius of gyration x can be assumed in practice to be the inner radius of the ring.

W = weight of a fly-wheel in pounds.
X = radius of gyration in feet.
n = revolutions per minute.
K = work in the fly-wheel in foot-pounds.
V = velocity of centre of gyration in feet per second.

$$K=\frac{MV^2}{2}=\frac{WV^2}{2\times32.17},$$

of which $$V^2=\left(\frac{2\pi Xn}{60}\right)^2\frac{2^2\pi^2}{2\times32.17\times60^2}=5867.16.$$

$$K=\frac{WX^2n^2}{5867.16}.$$

This formula gives the work of a fly-wheel or of any rotating body; that is to say, it requires that much work to bring a body from rest to a rotation of n revolutions per minute; or if the body is rotating with an angular velocity of n revolutions per minute, it will regenerate that much work before it is brought to rest.

$$K=\frac{WX^2n^2}{5867.16}. \quad . \quad . \quad 1 \qquad\qquad X=\frac{76.6}{n}\sqrt{\frac{K}{W}}. \quad . \quad . \quad 3$$

$$W=\frac{5867.16\,K}{X^2n^2}. \quad . \quad . \quad 2 \qquad\qquad n=\frac{76.6}{X}\sqrt{\frac{K}{W}}. \quad . \quad . \quad 4$$

Example 1. A fly-wheel of $W=2000$ pounds, and radius of gyration $X=3$ feet, is to be set in rotation by a weight $F=600$ pounds.

Fig. 162. What angular velocity will the fly-wheel obtain by the weight falling 50 feet?

The work $K = 600 \times 50 = 3000$ foot-pounds.

Revolutions $n = \frac{76.6}{3}\sqrt{\frac{3000}{2000}} = 31.278$ per minute.

In this case it makes no difference what radius the weight F is acting upon, because the weight multiplied by the fall will be the work stored in the fly-wheel; and the same work will be re-utilized in bringing the fly-wheel to rest.

Example 2. A fly-wheel weighing $W = 16000$ pounds, and radius of gyration $X = 6$ feet, is revolving at the rate of $n = 60$ revolutions per minute. Required the work in the wheel?

$$K = \frac{16000 \times 6^2 \times 60^2}{5867.16} = 353425 \text{ foot-pounds.}$$

The fly-wheel can wind up a weight of 353425 pounds one foot high, or 3534.25 pounds to a height of 100 feet. Whatever height divided into 353425 will be the weight the wheel can wind up to that height.

§ 136. FLY-WHEELS IN REGARD TO TIME AND SPACE.

In the preceding paragraph we treated fly-wheels in regard to work and angular velocity, without regard to the time and space which are constituent elements of work. We will now treat on the time required for storing or re-storing the work in a fly-wheel, and the space or total number of revolutions in the time required to bring the fly-wheel from rest to a uniform rotation or from a uniform rotation to rest.

N = total number of revolutions in the time T seconds.

The space of uniformly accelerated or retarded motions is

$$S = \frac{V\,T}{2},$$

when V means the uniform velocity of centre of gyration.

$$S = 2\,\pi\,X\,N = \frac{2\,\pi\,X n\;T}{2+60}.$$

$$n = \frac{120\,N}{T}.$$

$$\text{Work,}\quad K = \frac{W V^2}{2g} = \frac{W X^2 n^2}{5867.16}.$$

$$K = \frac{W X^2}{5867.16}\left(\frac{120 N}{T}\right)^2 = \frac{2.4543\, W X^2 N^2}{T^2}.$$

$$K = \frac{2.4543\, W X^2 N^2}{T^2}. \quad . \quad 1$$

$$W = \frac{K T^2}{2.4543\, X^2 N^2}. \quad . \quad 2$$

$$T = 1.5666\, X N \sqrt{\frac{W}{K}}. \quad . \quad 3$$

$$N = \frac{T}{1.5666\, X}\sqrt{\frac{K}{W}}. \quad . \quad 4$$

Example 1. A fly-wheel at rest weighing $W = 10000$ pounds is to be put into $n = 80$ revolutions per minute in $N = 6$ revolutions of the wheel. Required the time T? The radius of gyration being $X = 5$ feet.

$$\text{Work,}\quad K = \frac{10000 \times 5^2 \times 80^2}{5867.16} = 295504 \text{ foot-pounds.}$$

$$\text{Time,}\quad T = 1.5666 \times 5 \times 6 \sqrt{\frac{10000}{295504}} = 9.0986 \text{ seconds.}$$

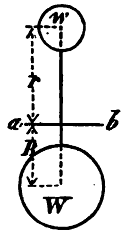
Example 2. The fly-wheel in the preceding example is to be stopped in $T = 5$ seconds. How many revolutions N will the wheel make in the action of stopping it?

$$\text{Revolution,}\quad N = \frac{5}{1.5666}\sqrt{\frac{295504}{10000}} = 3.3.$$

§ 137. ON THE CENTRING OF REVOLVING BODIES.

This subject has given much trouble to mechanical engineers, for the reason that a body perfectly balanced around its axis of rotation does not appear to be balanced when set into a high rotary velocity.

Fig. 164.



Let the bodies W and w be connected by an inflexible rod $R\ r$, and the system perfectly balanced on the line $a\ b$; that is,

$$W R = w\ r.$$

R and r are distances from the fulcrum $a\ b$ to the centres of gravity of their respective bodies.

Suppose the system to rotate in a plane at right angles to the axis $a\ b$, then the centrifugal force of each body will be

$$F = \frac{W R n^2}{29335}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 1$$

$$F = \frac{w\ r\ n^2}{2933.5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 2$$

But $W R = w\ r$, consequently the centrifugal forces of the bodies must be alike, and have no tendency to disturb the equilibrium in the fulcrum of rotation.

The work stored in each body by setting the system in rotation will be

$$K = \frac{W X^2 n^2}{5867.16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 3$$

$$K' = \frac{w\ x^2 n^2}{5867}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 4$$

in which X and x are the respective radii of gyration, but for simplicity in the illustration we can (without much error) consider $x = R$ and $x = r$, and we have the works,

$$K = \frac{W R^2 n^2}{5867.16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 5$$

$$K' = \frac{w\ r^2 n^2}{5867.16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 6$$

We have $W\ R = w\ r$, but $W R^2$ is not equal to $w\ r^2$, and consequently the works stored in eacn body are not alike. The small body w on the greater radius r has taken up more work than has the greater body W on the smaller radius R; and it is this difference of

work which caused an action in the fulcrum whilst the bodies were set in rotation. This irregular distribution of work in revolving bodies has puzzled many good mechanics.

Example. $W = 150$ pounds and $R = 0.5$ feet.
$w = 20$ pounds and $r = 3.75$ feet.
$n = 500$ revolutions per minute.

Required the centrifugal force? and work stored in each body?

Centrifugal force $F = \dfrac{150 \times 0.5 \times 500^2}{2933.5} = 6391.7$ pounds of each body.

The works concentrated in the bodies will be respectively

$$K = \frac{150 \times 0.5^2 \times 500^2}{5867} = 1598 \text{ foot-pounds.}$$

$$K' = \frac{20 \times 3.75^2 \times 500^2}{5867} = 11984.4 \text{ foot-pounds.}$$

We see here that the work stored in the small body w is over seven times greater than that stored in the larger body W; and it is this difference which causes the revolving system to work irregular in the fulcrum.

Suppose the two bodies W and w to be cast-iron balls, and find their diameter and radii of gyration? The diameters are nearly $D = 0.58$ and $d = 0.207$ of a foot.

$$\text{Radius gyration} \left\{ \begin{array}{l} X = \sqrt{R^2 + \dfrac{D^2}{10}} = \sqrt{0.5^2 + \dfrac{0.58^2}{10}} = 0.533 \text{ feet.} \\ x = \sqrt{r^2 + \dfrac{d^2}{10}} = \sqrt{3.75^2 + \dfrac{0.209^2}{10}} = 3.75 \text{ feet.} \end{array} \right.$$

The radius of gyration of the large ball is 0.033 of a foot longer than the radius R; but in the small ball there is no appreciable difference between x and r.

The real work stored in the large ball will then be

$$K = \frac{150 \times 0.533^2 \times 500^2}{5867} = 1815 \text{ foot-pounds.}$$

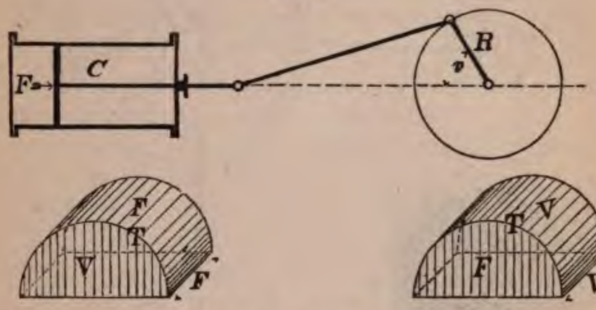
or $1815 - 1598 = 217$ foot-pounds more than in the first calculation.

Any system of revolving bodies which is balanced when at rest will also be balanced in any uniform rotation; but when the rotation is accelerated or retarded work is stored or re-stored, which causes an irregularity in the axis if the several radii of gyration are not balanced.

§ 138. TRANSFORMATION OF STRAIGHT LINEAR MOTION TO ROTARY BY THE AID OF A CRANK.

The illustration represents a steam-engine, of which the piston C has a straight linear motion, which is transformed into rotary by the

Fig. 165.



crank R. It is supposed that the irregular motion of the piston accommodates itself to the uniform rotary motion of the crank.

Let F denote the force acting constantly on the piston throughout the stroke; then the force acting on the crank R in the direction of the tangent of the circle will be $F \sin.v$. The volumes of the diagrams represent the works in the cylinder and crank respectively for a single stroke or half a revolution in the time T. The work in the cylinder is composed of a constant force by variable velocity, whilst that in the crank is composed of a variable force by constant velocity, but the two works or the products of the three simple $F\,V\,T$ are alike in both cases.

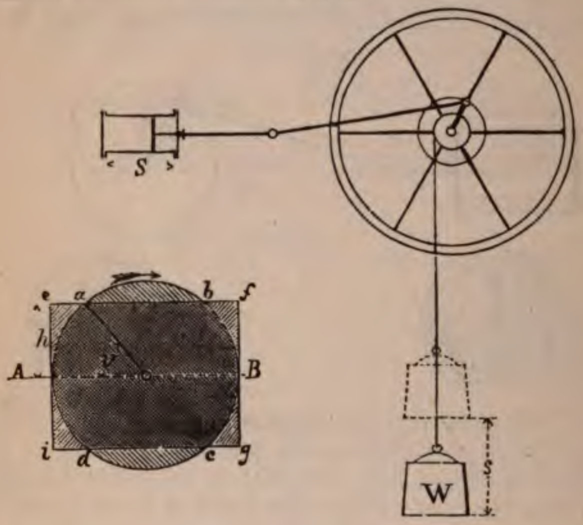
The variable velocity of the piston and the variable force of rotation are represented by the ordinates in the respective semicircles. The time is represented in this case by the length of the semicircles.

The force and velocity in one diagram take one another's place in the other diagram.

§ 139. ON THE REGULATION OF IRREGULAR WORK BY FLY-WHEELS.

Fig. 166 represents a steam-engine and fly-wheel winding up a weight, which operation can be compared with any uniform work accomplished by an engine. F = constant force acting on the piston throughout the stroke S.

Fig. 166.



Then $F : W = s : S$ for each single stroke or half a revolution of the wheel.

$$FS = Ws, \qquad F = \frac{Ws}{S}, \qquad W = \frac{FS}{s}, \qquad S = \frac{Ws}{F}, \qquad s = \frac{FS}{W}.$$

For simplicity in the illustration it is supposed that the connecting rod is infinitely long, that the steam-pressure is constant throughout the stroke of the piston, that all the work of the engine is transmitted through the crank-shaft and regulated by the fly-wheel to a uniform power in the realized work of rotation.

When the force F is constant throughout the stroke S, the work in the cylinder can be represented by the area of a circle $A\ a\ b\ B\ c\ d$ for one revolution of the engine. The area of the semicircle $A\ b\ c\ B$ will then represent the work for a single stroke of the piston. The diameter $A\ B$ is equal to the stroke S of the piston, and is sup-

posed to be in the centre line of the engine, so that the crank-pin describes the circle and is on the centre at A and B.

When the crank passes the centres A and B the engine performs no work in the rotary motion, and when the power in the realized work is constant in all positions of the crank, the fly-wheel must perform the full power of the engine when the crank passes the centres. The power of the engine varies as the *sine* for the angle of the crank to the centre line $A\ D$, and the work performed in the time of one revolution of the crank is represented by the area of the circle described.

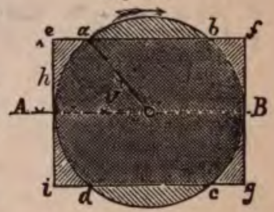
Draw on $A\ B$ the two rectangles $A\ e\ f\ B$ and $B\ g\ i\ A$, equal to the area of the circle; then the height $h = A\ e$ represents the mean velocity of the piston throughout the stroke S. Suppose the engine to be running with a uniform rotation of the fly-wheel, the area of the semicircle will then represent the real work, and the rectangle the mean work on the piston for a single stroke.

It will be seen that the circle projects over the rectangle with a segment $a\ b$, which is the work stored in the fly-wheel during that portion of the stroke, or whilst the crank-pin passes from a to b. The area of the corners $b\ f\ B$ and $B\ g\ c$ of the rectangles which project over the circle are equal to the area of the segment $a\ b$, or the work restored to the rotary motion by the fly-wheel, whilst the crank passes from b via centre B to c, and another segment of work $c\ d$ is stored in the fly-wheel and restored to the work of rotation whilst the crank passes from d via centre A to a. The fly-wheel is thus regulating the irregular work in the steam cylinder to a uniform work of rotation.

§ 140. TO FIND THE IRREGULARITY OF ROTATION OF A FLY-WHEEL.

The alternate storing and re-storing of work by a fly-wheel causes a slight irregularity of rotation.

Fig. 167.



The greater the fly-wheel is and the greater the velocity of rotation, the less will the irregularity be.

Suppose the crank and fly-wheel to move in the direction indicated by the arrow, and the rotation will be fastest when the crank-pin passes b and d, and slowest at a and c.

Let R denote the radius of the crank. The area or work of the semicircle $A\ a\ b\ B$ will be $\frac{1}{2}\ \pi\ R^2$. The area or mean work of the rectangle $A\ e\ f\ B$ will be $2\ R\ h$.

Then $\frac{1}{2}\pi R^2 = 2Rh$, of which $h = \frac{\pi R}{4} = 0.7853\ R.$

That is to say, the mean velocity of the piston is equal to 0.7853 of the circular velocity of the crank-pin. The fraction 0.7853 is the *sine* for the angle v of the crank with the centre line of the engine when the rotary motion is a maximum at b and d, and minimum at a and c. The angle $v = 51°\ 45'$.

The work k stored and re-stored alternately by the fly-wheel or the area of the segment $a\ b$ will be

$$k = \frac{\pi R^2(90 - v)}{180} - R^2 \sin.v \cos.v.$$

$$v = 51.75°. \qquad \sin.v = 0.7853. \qquad \cos.v = 0.61909.$$

$$k = \frac{3.14 \times 38.25 R^2}{180} - 0.7853 \times 0.61909\ R^2 = 0.18109\ R^2.$$

This is the area of the segment a, b expressed in a fraction of a square radius; but we want the segment of work to be expressed in a fraction of the semicircle, or of the work of a single stroke of the piston.

The area of a semicircle $= \frac{1}{2}\pi\ R^2$.

Call $FS =$ the work of a single stroke,

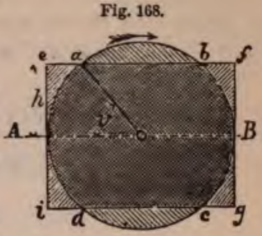
$$\text{Then} \quad k = \frac{0.18109\ R^2}{\frac{1}{2}\pi\ R^2} = 0.1153\ FS.$$

That is to say, the irregular work stored and re-stored by a fly-wheel is equal to 0.1153 of the work of a single stroke of the piston. As the same proportion is constant for every stroke of the engine, it follows that the work of regulation by a fly-wheel is always 0.1153 of the work of the engine, when the steam-cylinder is double acting and the realized work is uniform.

§ 141. ON IRREGULARITY OF ROTATION OF A FLY-WHEEL.

Notation of Letters (repeated).

Fig. 168.



W = weight of fly-wheel in pounds.
x = radius of gyration in feet.
n = number of revolutions per minute.
K = work in foot-pounds in the fly-wheel.
f = irregularity in fraction of the revolutions n, or of the mean velocity of rotation.

The mean velocity of the fly-wheel is when the crank passes the centres at A and B, the greatest velocity at b and d, and the slowest at a and c. Whilst the crank-pin passes from A to a the work represented by the projecting corner $A\,e\,a$ is given out from the fly-wheel, by which the velocity is reduced.

The work of the corner $A\,e\,a = \frac{1}{2} \times 0.1153\,FS = 0.0576\,FS.$

The work in the fly-wheel is $K = \dfrac{W x^2 n^2}{5867.16}$.

Mean revolutions $n = \dfrac{76.6}{x}\sqrt{\dfrac{K}{W}}$.

Revolutions or angular velocity at a,

$$n' = \frac{76.6}{x}\sqrt{\frac{K - 0.0576\,FS}{W}}.$$

$$\text{Irregularity} \quad f = \frac{n - n'}{n} = 1 - \sqrt{1 - \frac{0.0576\,FS}{K}}.$$

$$f = 1 - \sqrt{1 - \frac{338\,FS}{W x^2 n^2}}. \quad . \quad 1$$

$$W = \frac{338\,FS}{x^2 n^2 [1 - (1 - f)^2]}. \quad . \quad 2$$

$$n = \frac{18.4}{x}\sqrt{\frac{FS}{W[1 - (1 - f)^2]}}. \quad 3$$

$$x = \frac{18.4}{n}\sqrt{\frac{FS}{W[1 - (1 - f)^2]}}. \quad 4$$

Example. A fly-wheel of $W = 12000$ pounds, and radius of gyration $x = 7$ feet, is to make $n = 48$ revolutions per minute, with an engine of $S = 4$ feet stroke and steam-pressure on the piston $F = 12750$ pounds. Required the irregularity of rotation of the fly-wheel?

$$f = 1 - \sqrt{1 - \frac{338 \times 12750 \times 4}{12000 \times 7^2 \times 48^2}} = 0.00639.$$

This very small irregularity could not be perceived without a delicate dynamometer, and the fly-wheel is sufficiently large.

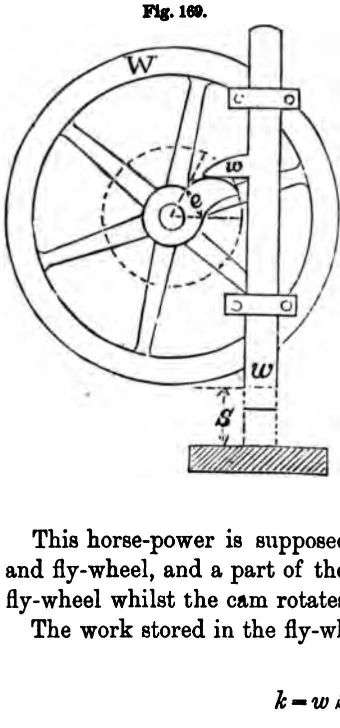
The irregularity should never exceed 0.1, and in ordinary practice f should be from 0.01 to 0.05.

When $f=1$ the fly-wheel cannot carry the engine over the centre. $f=1$, when $338\,FS=Wx^2n^2$.

§ 142. FLY-WHEEL FOR A STAMP-MILL.

Notation of Letters.

Fig. 169.



W = weight of fly-wheel in pounds.
w = weight of stamp in pounds.
S = vertical height in feet which the stamp is lifted.
n = number of blows per minute.
e = angle which the cam rotates whilst lifting the stamp.
HP = horse-power in operation.

The work done by each blow will be

$$K=wS. \quad . \quad 1$$

Per minute $\quad K=wSn. \quad . \quad 2$

Horse-power $\quad \mathrm{HP}=\frac{wSn}{33000}. \quad . \quad 3$

This horse-power is supposed to be constant in driving the shaft and fly-wheel, and a part of the work of that power is stored in the fly-wheel whilst the cam rotates freely without lifting the stamp.

The work stored in the fly-wheel between each lift is

$$k=wS\left(\frac{360^\circ-e}{360^\circ}\right). \quad . \quad . \quad . \quad . \quad 4$$

which work is re-utilized in lifting the stamp.

N = rate of revolutions per minute at the moment the cam commences to lift the stamp, when the work stored in the fly-wheel is

$$K=\frac{Wx^2N^2}{5867.16}. \quad . \quad . \quad . \quad . \quad 5$$

u = rate of revolutions at the moment the cam drops the stamp, when the work stored in the fly-wheel is

$$K = \frac{W x^2 u}{5867.16}. \quad . \quad . \quad . \quad . \quad 6$$

$$K - k = w\,S\left(\frac{360 - e}{360}\right) = \frac{W x^2}{5867.16}(N^2 - u^2).$$

$$(N^2 - u^2) = \frac{5867.16\,w\,S}{W x^2}\left(\frac{360 - e}{360}\right).$$

$$u = \sqrt{N^2 - \frac{5867.16\,w\,S}{W x^2}\left(\frac{360 - e}{360}\right)}. \quad . \quad . \quad . \quad 7$$

The irregularity of rotation of the fly-wheel will then be

$$f = \frac{N - u}{N} = 1 - \sqrt{1 - \frac{5867.16\,w\,S}{W x^2 u^2}\left(\frac{360 - e}{360}\right)}. \quad . \quad . \quad 8$$

Example. A stamp weighing $w = 1200$ pounds is to be lifted $S = 3$ feet $n = 48$ times per minute. The weight $W = 9720$ pounds and radius gyration $x = 5.25$ feet; the cam moves $e = 120°$ in lifting the stamp. Required the irregularity f of the rotation of the wheel?

$$f = 1 - \sqrt{1 - \frac{5867.16 \times 1200 \times 3}{9720 \times 5.25^2 \times 48^2}\left(\frac{360 - 120}{360}\right)} = 0.01147.$$

This irregularity is very small, and the fly-wheel is consequently sufficiently large, and could be made a little smaller.

The work consumed direct from the motive-power in raising the stamp is

$$K = w\,S\left(\frac{e}{360}\right). \quad . \quad . \quad . \quad . \quad 9$$

and the work taken out from the fly-wheel is

$$K = w\,S\left(\frac{360 - e}{360}\right). \quad . \quad . \quad . \quad . \quad 10$$

The total work consumed in raising the stamp is

$$K = w\,S\left(\frac{e}{360}\right) + w\,S\left(\frac{360 - e}{360}\right) = w\,S\left(\frac{e}{360}\right)\left(\frac{360 - e}{360}\right).$$

$$K = w\,S\left(\frac{360 - e + e}{360}\right) = w\,S, \text{ the realized work.} \quad . \quad 11$$

ELEMENTS OF FLY-WHEELS.

Outside diameter of wheel.	Square section of ring.	Radius of gyration.	Weight of wheel.	Work $K = Cn^2$	Mass M
feet.	inches.	x in feet.	W in lbs.	coef. C	matts.
2	2	0.82	120	0.0138	3.7202
2	2.5	0.80	180	0.0196	5.5853
2	3	0.76	270	0.0266	8.3930
2.5	2.5	1.03	234	0.0424	7.2739
2.5	3	1.00	336	0.0625	10.444
2.5	3.5	0.95	460	0.0710	14.300
3	2.5	1.30	281	0.081	87.347
3	3	1.20	405	0.100	12.589
3	3.5	1.00	550	0.0937	17.095
3.5	3	1.50	472	0.180	14.673
3.5	3.5	1.30	640	0.184	19.895
3.5	4	1.10	840	0.174	26.112
4	3	1.75	540	0.282	16.784
4	4	1.42	960	0.330	29.842
5	3	2.25	675	0.582	20.973
5	4	2.17	1,200	0.960	37.202
6	4	2.66	1,440	1.750	44.763
6	5	2.60	2,250	2.600	81.721
8	5	3.60	3,000	6.63	93.255
8	6	3.50	4,320	9.03	134.29
10	6	4.50	5,400	18.65	167.84
10	8	4.20	9,600	28.9	298.42
12	6	5.50	6,480	33.4	201.43
12	9	5.25	9,720	45.8	302.15
16	9	7.25	12,960	116.5	402.86
16	12	7.00	34,560	289	1074.3
20	9	9.25	24,300	355	755.37
20	12	9.00	43,200	598	1342.9
25	12	11.50	64,800	1460	2014.3
25	15	11.25	84,300	1820	2620.5
30	12	14.00	65,800	2200	2045.4
30	15	13.75	100,250	3230	3116.3
36	15	16.75	120,500	5740	3735.7
36	18	16.5	170,000	7500	5274.4

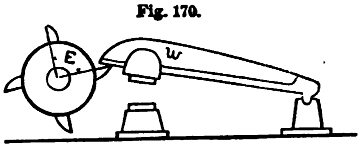
The weight of cast-iron fly-wheels is closely approximated by the following rule:

Multiply the cross-section of the ring in square inches by the outside diameter of the wheel in feet, and the product by 15, which will be the weight in pounds of the fly-wheel, including arms and centre.

The square of the number of revolutions per minute multiplied by the coefficient in the column C will be the work in the wheel.

§ 143. FORGE-HAMMER WORKED BY CAMS.

Fig. 170.



The illustration presents a forge-hammer worked by four cams. The fly-wheel is on the shaft *c*.

Notation of Letters.

W = weight of fly-wheel in pounds.

w = weight of the hammer in pounds, including its whole moving system.

S = vertical space in feet which the centre of gravity of the hammer is lifted.

n = number of revolutions of the fly-wheel per minute.

N = number of lifting cams in the circle.

E = angle in degrees between each cam.

e = angle which the cams rotate whilst lifting the hammer.

HP = horse-power in operation.

x = radius of gyration of the fly-wheel.

f = irregularity of rotation of the fly-wheel between each blow of the hammer.

The work done in each blow of the hammer is

$$K = w\,S. \qquad . \quad . \quad . \quad . \; 12$$

Per minute $$K = w\,S\,N\,n. \qquad . \quad . \quad . \; 13$$

Horse-power $$\text{HP} = \frac{w\,S\,N\,n}{33.000}. \qquad . \quad . \quad . \; 14$$

This is the horse-power required to drive the hammer in making $N\,n$ blows per minute.

The work stored in the fly-wheel between each blow is

$$k = w\,S\left(\frac{E-e}{E}\right), \qquad . \quad . \quad . \quad . \; 15$$

which work is re-utilized in lifting the hammer.

The total work stored in the fly-wheel is

$$K' = \frac{W\,x^2\,n^2}{5867.16}. \qquad . \quad . \quad . \quad . \; 16$$

The irregularity of work between each lift of the hammer will then be

$$k' = WS\left(\frac{E-e}{E}\right)\frac{5867.16}{WX^2n^2}. \quad . \quad . \quad . \quad 17$$

The irregularity of rotation of the fly-wheel between each blow of the hammer will be

$$f = \sqrt{k'} = \frac{76.6}{Xn}\sqrt{\frac{wS}{W}\left(\frac{E-e}{E}\right)}. \quad . \quad . \quad . \quad 18$$

Example 1. A hammer like that represented by the illustration, weighing $w = 12000$ pounds, is worked with $N = 4$ cams making $n = 10$ revolutions per minute, and the centre of gravity of the hammer is lifted $S = 1$ foot each blow. Four cams make the angle $E = 90°$ and the angle $e = 30°$. The weight of the fly-wheel is $W = 80000$ pounds. Required the irregularity of the fly-wheel for each blow of the hammer? Radius of gyration of the fly-wheel is $X = 12$ feet.

$$f = \frac{76.6}{12 \times 10}\sqrt{\frac{12000 \times 1}{80000}\left(\frac{90-30}{90}\right)} = 0.2,$$

the irregularity of the fly-wheel, which is rather too much.

It is supposed in this example that the fly-wheel is on the same shaft as the cams, but if placed on another shaft and geared to make more revolutions than the cam-shaft, the irregularity would be reduced in proportion to the gearing, as the formula will give by inserting for n whatever revolutions the fly-wheel makes per minute.

§ 144. ILLUSTRATION OF IRREGULAR ROTATION OF A STEAM-ENGINE CRANK WITH FLY-WHEEL.

Fig. 171.

Let A B represent the centre line of a steam-engine and C D at right angles thereto. R = radius of the crank, which will be on the centre at A and B, and at right angles to the engine in the points C and D.

Let the circle A C B D described by the crank-pin represent the mean angular velocity, and the ellipse a b c d the irregularity of rotation; that is, the distance from the centre to the periphery of the ellipse represents the velocity of rotation in that position of the crank, which is slowest at a and c, and fastest at b and d.

The greatest distance between the circle and ellipse at the points a b c and d is the irregularity f corresponding to the preceding formulas.

It is supposed here that the steam-pressure on the piston is constant throughout the stroke, and that the realized work is uniform in the rotation.

When the steam is expanded half-stroke the ellipse will be nearer the circle.

§ 145. TO APPROXIMATE THE SIZE AND WEIGHT OF A FLY-WHEEL for one double-acting steam-cylinder when the realized work is uniform.

Assume that the ring of the fly-wheel is of a square section, that the radius of gyration is equal to the inner radius of the ring, and that the weight of all the arms and of the hub is equal to half the weight of the ring. Assume also the side of the square section of the ring to be one inch for each foot of the radius of gyration, it follows that the weight of the fly-wheel will be

$$W = 2\pi\left(X + \frac{X}{24}\right)\left(\frac{X}{12}\right)^2 \times 450 \times 1\tfrac{1}{2}. \quad . \quad . \quad . \quad 1$$

$$W = \frac{X^3 \times 2\pi \times 1.04166 \times 450 \times 1\frac{1}{2}}{144} = 30\,X^3. \quad . \quad . \quad 2$$

$$W = \frac{338\,FS}{X^2 n^2[1-(1-f)^2]} = 30\,x^3. \quad . \quad . \quad . \quad . \quad 3$$

We may limit the irregularity of the wheel to $f = 0.025$;

then $\quad [1-(1-0.025)^2] = 0.05,$

and $\quad 30\,X^5 = \dfrac{338\,FS}{0.05\,n^2}, \quad$ and $\quad X^5 = \dfrac{338\,FS}{1.5\,n^2}.$

$$X = 3\sqrt[5]{\frac{FS}{n^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 4$$

Example. The cylinder of a steam-engine is 24 inches diameter by $S = 4$ feet stroke. Steam-pressure 50 pounds to the square inch; area of piston 452.39 square inches × 50 pounds = 22619.5 pounds = F the force on the piston. The engine to make $n = 60$ revolutions per minute. Required the size and weight of the fly-wheel?

$$\text{Radius gyration,} \quad X = 3\sqrt[5]{\frac{22619.5 \times 4}{60^2}} = 5.716,$$

say 6 feet the inside radius of the ring.

The side of the section of the ring should be 6 inches, and the outside diameter of the wheel will be $2 \times 6 + 1 = 13$ feet.

Weight of the fly-wheel $= 30 \times 6^3 = 6480$ pounds.

§ 146. FLY-WHEEL FOR TWO ENGINES CONNECTED AT RIGHT ANGLES.

When two engines are connected by cranks at right angles on a shaft, each cylinder conveys a work like that represented by the preceding diagrams.

Fig. 172.

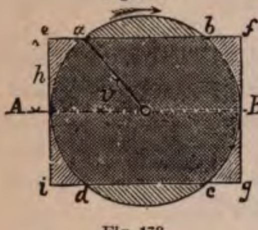


Fig. 173.

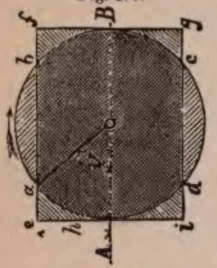
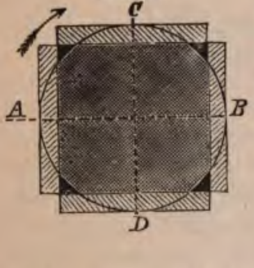


Fig. 174.



The one cylinder alone produces this diagram of work, Fig. 172, as before described. The centre-line being in the direction $A\ B$.

The other engine, connected by a crank at right angles to that of the former, produces a diagram like Fig. 173. The centre-line $A\ B$ of the engine being at right angles to that of the former.

Now place the two diagrams on the top of one another, and we have an illustration, Fig. 174, of the joint action of the two engines.

It will be seen that the segments of work which were stored and re-stored by the fly-wheel for each alternate stroke of one engine alone, are now utilized direct to the rotary motion; and the small irregularity of rotation is represented by the black corners outside of the circle; which shows that the crank will move a little faster when at angles of 45° on each side from the centre-line, and slowest when passing the centres, supposing that no fly-wheel is employed; but with a fly-wheel the rotation will be fastest when the crank passes the black points, which are at angles of 51° 45′, and slowest when the cranks pass 5° 15′ from their respective centre-lines.

§ 147. WHEN FLY-WHEELS ARE AND ARE NOT NECESSARY.

The duty of a fly-wheel is to regulate irregular work, which may be either or both the primitive or realized works. We have heretofore treated the fly-wheel as regulating the irregular primitive work in the steam-cylinder, to regular realized work of rotation.

In the case of a steam-engine lifting a hammer or a stamp by a cam, both the primitive and realized works are irregular.

When a water-wheel is lifting a hammer or a stamp by a cam, the primitive work is regular and the realized irregular.

In the case of a rolling-mill worked by a steam-engine, the irregularity of the primitive work is so small compared with that of the realized work in the mill, that the work of the fly-wheel must be proportionate entirely to that of the mill.

In the cases of pumping- and blowing-engines, where the primitive work is applied direct to the realized work, there is no work performed at the ends of the stroke, except that of moving valves and in overcoming friction; but when two such engines are connected by cranks at right angles on a shaft, the one engine will move the valves for the other one, and no fly-wheel is there required; in fact, such arrangements will work much better without a fly-wheel, like in marine engines.

In upright blowing-engines the heavy moving parts require a fly-wheel for regulating the motion, but if the weight of the moving parts are balanced, which can be done by steam, the fly-wheel would be of no utility, when two engines are connected at right angles.

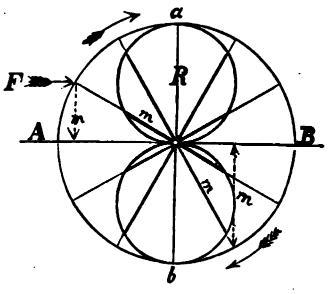
In horizontal blowing- and pumping-engines the moving parts are well balanced and require no fly-wheel.

When the steam is expanded in the cylinders, the one engine will work with full steam whilst that of the other is expanding, and they thus help one another alternately without fly-wheel.

§ 148. ROTARY MOMENTUM IN CRANK MOTION.

Let the circle $A\ a\ B\ b$ represent that described by a crank of radius R, and $A\ B$ being the centre line of the steam-engine. For simplicity in the illustration we will suppose the connecting-rod to be infinitely long, when the force F will act on the crank-pin in a direction parallel to that of the centre line $A\ B$, and the static momentum in any position of the crank will be the product of the force F and the sine (m) for the angle of the crank to the line $A\ B$.

Fig. 175.



Set off the sine m from the centre on its radius for any number of positions of the crank, and join the outer ends, which forms the curve of *rotary momentum*. When the connecting-rod is infinitely long, like in the case of slot-crank motion, the curve of rotary momentum

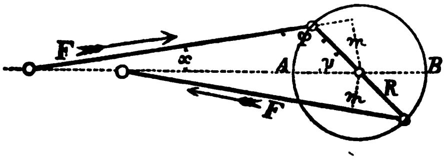
19

will be two circles inscribed in the circle of the crank, as shown by the illustration.

§ 149. ENGINE WITH CONNECTING-ROD OF DEFINITE LENGTH.

When the connecting-rod is of definite length the momentum of rotation will not be as the *sine* of the angle of the crank, but as the rightangular distance from the centre to the direction of the connecting-rod.

Fig. 176.



l = length of the connecting-rod.
R = radius of the crank.
v = angle of the crank.
x = angle of the connecting-rod.
F = force on the piston in pounds.
P = pressure in the guides.
d = distance from centre of crosshead to centre of crank-shaft.

$$l : R = \sin.v : \sin.x. \qquad \sin.x = \frac{R \sin.v}{l}.$$

$$d : l = \sin.\varphi : \sin.v. \qquad d = \frac{l \sin.\varphi}{\sin.v.} = \frac{l \sin.(180 - x - v)}{\sin.v.}.$$

The lever m for the momentum of the force F will then be

$$m = d \sin.x = \frac{l \sin.(180 - x - v) \sin.x}{\sin.v}.$$

The lever m can thus be calculated or constructed graphically for different positions of the crank. The lever m is set off from the centre on the corresponding position of the crank which forms the curve of rotary momentum.

The pressure in the guides will be

$$P = F \tan.x.$$

§ 150. MOMENTUM-CURVE FOR A SINGLE ENGINE with connecting-rod twice the stroke.

Fig. 177.

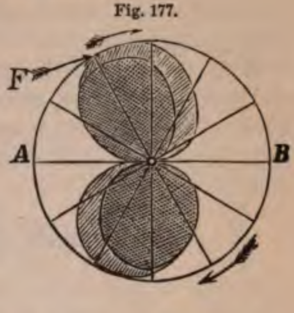
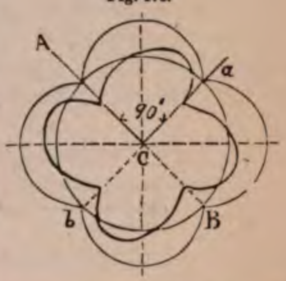


Fig. 177 shows the curve of rotary momentum when the length of the connecting-rod is twice that of the stroke. The shorter the connecting-rod is the more do the curves lean toward the engine. The outer curves show the rotary momentum when the steam-pressure is constant throughout the stroke, and the inner curves that when the steam is expanded $\frac{2}{3}$ of the stroke.

§ 151. MOMENTUM-CURVE FOR TWO ENGINES WORKING at right angles and connecting-rods infinitely long.

Fig. 178.



Curves of rotary momentum for two engines working at right angles on one common crank, which is the same as when two engines are working on cranks at right angles on a common shaft. The connecting-rods are supposed to be infinitely long, like when a crank-pin works in a slot. The circle *A a B b* is that described by the crank-pin around the centre *C*. The momentums for each engine are added for each position of the crank, for which the curves of rotary momentum extend outside of the circle of rotation. The outer curves are for full steam, and the inner ones for $\frac{2}{3}$ expansion.

§ 152. MOMENTUM-CURVE FOR TWO ENGINES WORKING at right angles and connecting-rods twice the stroke.

Fig. 179.

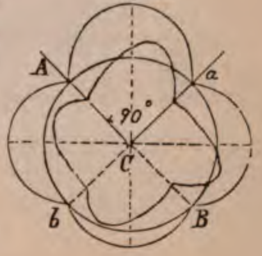


Fig. 179 shows the rotary momentum for two engines working at right angles on one crank, and the connecting-rod being twice the stroke.

The inner curve shows the rotary momentum when the steam is cut off at one-third the stroke, or with two-thirds expansion.

The curves of rotary momentum serve

to show the distribution of power and work in the circle described by the crank-pin.

In the illustrations of rotary momentum it is supposed that all the primitive work is transmitted through the cranks and shaft, which is most generally the case in steam-engines; but when the primitive work is transmitted directly to the realized work, like in blowing and pumping-engines, where the steam-piston is connected directly by a rod to the pumping piston, only a small portion of the work is transmitted alternately through the crank and shaft as rotary momentum.

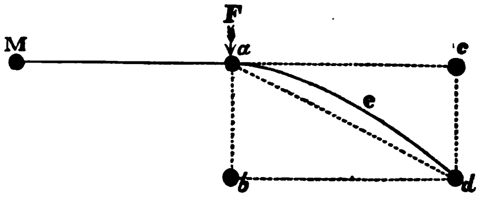
§ 153. CHANGING DIRECTION OF A MOVING MASS.

A free moving mass cannot within itself change its direction of straight linear motion.

The direction of a moving mass can be changed only by applying a force in a different direction.

A mass *M* is moving with a uniform velocity in the direction *a c*, but at *a* a constant force *F* is applied at right angles to *a c*, so that the force would move the mass from *a* to *b* in the same time as the mass of itself would move from *a* to *c*.

Fig. 180



The resultant of the two motions will carry the mass from *a* to *d* in that same time. As the force *F* is constant, the motion in the direction *a b* or *c d* will not be uniform, but accelerated in accordance with the laws of dynamics of matter—namely, that the spaces passed through by a mass acted upon by a constant force are as the square of the times of action. Therefore, the mass *M* will not move in a straight line from *a* to *d*, but in the course *a*, *e*, *d*, which is a parabola.

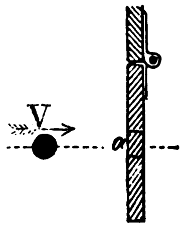
A jet of water flowing from a vertical orifice shows this parabolic course, in which case the force *F* is the force of gravity, the mass *M* is the water which is set in motion by the pressure from behind the orifice.

§ 154. FIRING A MUSKET-BALL THROUGH A DOOR.

Fig. 181 represents a horizontal section of a door movable on hinges.

A musket-ball V is fired so as to penetrate the door at a. It has been found by experiment that it requires a force of 100 pounds to force the ball slowly through the door one inch thick.

Fig. 181.



The work of penetration is then $\frac{100 \times 1}{12} = 8.33$ foot-pounds.

$W = 0.071$ of a pound, the weight of the ball.

$V = 1000$ feet per second, velocity of the ball.

v = velocity after penetration.

The work K stored in the ball before striking is

$$K = \frac{W V^2}{2g} = \frac{0.071 \times 1000^2}{2 \times 32.17} = 1103.6 \text{ foot-pounds.}$$

After penetration the work is $k = 1103.6 - 8.3 = 1095$ foot-pounds. Velocity v after penetration will be

$$v = \sqrt{\frac{k g}{W}} = \sqrt{\frac{1095 \times 32.17}{0.071}} = 704.4 \text{ feet per second.}$$

The loss of velocity by penetration is 295.6 feet per second.

The mean velocity through the door was 852 feet per second.

The time of penetration was

$$T = \frac{1}{852 \times 12} = 0.0000097844 \text{ of a second.}$$

The door is of white pine, 3 feet wide by 6 feet high, which will weigh 52 pounds. Suppose the door to move freely without friction in the hinges and without resistance of air. The ball penetrates the door in the centre, or at $r = 1.5$ feet from the line of the hinges.

Radius of gyration of the door is $X = 3 \times 0.5775 = 1.7325$ feet.

In accordance with the assumed conditions the door will be set into rotation by the penetration of the ball.

From § 112 we have the number of revolutions per minute

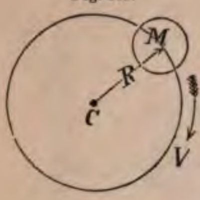
$$n = \frac{60\, F r g T}{2 \pi W X^2} = \frac{60 \times 100 \times 1.5 \times 0.0000097844}{6.28 \times 0.071 \times 1.7325^2} = 0.065851$$

of one revolution, or it would require 15.1 minutes for the door to make only one revolution, which motion would hardly be perceptible, but the resistance of air and friction in the hinges would keep the door almost stationary.

§ 155. CENTRIFUGAL AND CENTRIPETAL FORCES.

A mass M moving with a velocity V in the circle of radius R around the centre C will have a tendency to move outward from the centre, for which a force equal to that tendency must be applied on the mass from the centre C, to keep it revolving in the circle.

Fig. 182.



The tendency of the mass to move outward is called *centrifugal force*, and the force resisting that tendency is called *centripetal force*. These two forces are equal and in opposite directions; neither one of them can exist without the other, in fact, their distinction is only action and reaction, the only condition under which force can be conceived or realized.

Tie a stone at the end of a string and swing it round in the air, and a force is felt in the hand which you may call centrifugal or centripetal as you please. The tension of the string is the centrifugal force, and the reaction by the hand is the centripetal force.

§ 156. THEORY OF CENTRIFUGAL FORCE.

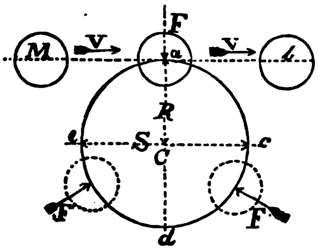
Let a mass M, Fig. 180, move with a uniform velocity V in the direction a, b until it arrives at a, where a constant force F is applied on it at right angles to the motion, which causes the body to deviate from the straight line a, b; but as that force acts at right angles to the motion, it has no effect on the velocity V of the mass. Suppose the direction of the force F to vary with the deviation of the motion from the straight line, so as to always be at right angles to the direction of motion; it follows that the velocity of the mass will be constant. When the force F is so balanced as to cause the mass to move in a circle of radius R around the centre C, the mass will then return to the same point a where the constant force was first applied. If the force F ceases to act at the moment it returns to a, the mass will continue with the same velocity in its original course a, b, which is the tangent to the circle. Therefore, when the centripetal force of a revolving body ceases to act, the body will fly off in the direction of the tangent to that point of the circle where it was let loose.

Although the velocity of the mass was not changed in its circular motion, the force F actually stopped the mass at c, giving it a backward motion at d, stopped it at e and returned it into a forward motion again with the same velocity V at a, all in reference to the

direction a, b; which operation was accomplished in the space of the diameter of the circle.

It follows that F is equal to a force which would give the mass M a velocity V in a space equal to the diameter of the circle, and which is the centrifugal force of the mass.

Fig. 183.



We have $F : M = V : T$, $F = \frac{M\,V}{T}$.

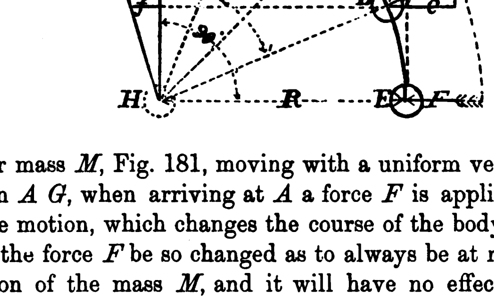
Diameter $2\,R = \frac{V\,T}{2}$, and $T = \frac{R}{V}$.

Centrifugal force $F = \frac{M\,V^2}{R}$.

F = force in pounds, M = mass expressed in matts., V = velocity in feet per second of the mass in the circle of radius R in feet.

§ 157. PROOF OF CENTRIFUGAL FORCE.

Fig. 184.



A body or mass M, Fig. 181, moving with a uniform velocity V in the direction $A\ G$, when arriving at A a force F is applied at right angles to the motion, which changes the course of the body. Let the direction of the force F be so changed as to always be at right angles to the motion of the mass M, and it will have no effect upon the velocity V in the path of motion. The force F can be of such magnitude as to cause the body to describe a circle A, B, C, D, F of radius R. Having given the mass M, velocity V and radius R, the problem is to find the magnitude of the force F.

When the body has arrived at B, the force F acting toward the centre H can be resolved into two forces a and b, of which a acts to stop the motion in the direction parallel to A, G, and b acts to set the body in motion in the direction parallel to A, H.

$$a = F \sin.x. \quad \text{and} \quad b = F \cos.x.$$

In the position C we have $c = F \sin.\varphi$, and $d = F \cos.\varphi$.

At D, $e = F \sin.z$, and $f = F \cos.z$.

When the body arrives at E, its motion in the direction parallel to A G has been stopped by the forces a, c, e, and finally F. If the variable force which stopped the body in the direction A G had been applied opposite the motion at A, it would have stopped the body at G. Set off the forces a, c, e and F at right angles to A G in the respective positions of the body. Join these forces with A and m, which will be a straight line. Then the area of the triangle A, m, G represents the

Work $k = \frac{F R}{2}$, consumed in stopping the body in the direction A G, which must be equal to the work $k' = \frac{M V^2}{2}$ stored in the body when arriving at A, or

$$\frac{F R}{2} = \frac{M V^2}{2}, \text{ of which the centripetal force } F = \frac{M V^2}{R},$$

and which is the solution of the problem.

The forces b, d and f, acting in the direction parallel to A H, have given the body the velocity V when arriving at E, or if that variable force had been applied on the body at rest at A in the direction A H, it would have produced the velocity V at H. Set off the forces F, b, d and f at right angles to the line A H, and at the respective positions of the body. Join these forces with n and H, which will be a straight line. The area of the triangle A n H represents the

$$\text{Work} \quad k = \frac{F R}{2}$$

accomplished by the variable force acting in the direction parallel to A H, and which is equal to the

$$\text{Work} \quad k' = \frac{M V^2}{2}$$

stored in the body when arriving at E or H.

The centripetal force is equal to the centrifugal force of the revolving body, or the force of inertia presented to the change of direction of motion, is equal to the centrifugal force $F=\frac{M V^2}{R}$.

It would appear from this formula that the centrifugal force is not a simple element, but it will be evident by the illustration that the spaces $A\ G$ or $A\ H$, which is the radius of the circle, is accomplished by the product of velocity and time, or $R=v\ T$, in which v is the mean velocity in the time T or space R, and the work $M\ V^2$ is divided by the space R, which gives the simple element force. When the mass is expressed by weight the centrifugal force will be

$$F=\frac{W V^2}{g R}. \quad \ldots \quad 1$$

The centrifugal force F is to the weight W of the revolving mass, as double the height of fall due to the velocity V is to the radius R of revolution. Call S = space of fall due to V, then

$$F : W = 2\ S : R, \quad \text{of which} \quad S=\frac{V^2}{2g}.$$

$$F : W = \frac{V^2}{g} : R. \qquad F=\frac{W V^2}{g R}, \text{ as before proved.}$$

Velocity $$V=\frac{2 \pi R n}{60}=0.10472\ R\ n. \quad \ldots \quad 3$$

Centrifugal force $$F=\frac{W}{g R}\left(\frac{2 \pi R n}{60}\right)^2=\frac{W R n^2}{2933.5}. \quad \ldots \quad 4$$

Weight of body $$W=\frac{2933.5\ F}{R n^2}. \quad \ldots \quad 5$$

Radius of revolution $$R=\frac{2933.5\ F}{W n^2}. \quad \ldots \quad 6$$

Number of revolutions $$n=\sqrt{\frac{2933.5\ F}{W R}}. \quad \ldots \quad 7$$

Velocity $$V=\sqrt{\frac{g R F}{W}}. \quad \ldots \quad 8$$

T = time in seconds of one revolution.

$$F=\frac{W R\ 60^2}{2933.5\ T^2}=\frac{1.2272\ W R}{T^2}. \quad \ldots \quad 9$$

Example 1. A body weighing $W=160$ pounds is revolving at the rate of $n=120$ revolutions per minute on a radius $R=1.5$ feet from centre of rotation to centre of gravity of the body (not centre of radius of gyration). Required the centrifugal force?

$$F=\frac{160\times1.5\times120^2}{2933.5}=1178 \text{ pounds.}$$

Example 2. The radius of the earth at the equator is about $R=20{,}900{,}000$ feet, and it makes one revolution in 24 hours $=1440$ minutes $=86400$ seconds $=T$. Required the centrifugal force of a mass at the equator weighing $W=2240$ pounds or one ton?

$$F=\frac{1.2272\times2240\times20{,}900{,}000}{86400^2}=7.6963 \text{ pounds.}$$

The radius of the earth in any latitude L is

$$R=20887680(1+0.00164\ \cos.2L).$$

The centrifugal force at any point on the surface of the earth will be

$$F=\frac{W(1+0.00164\cos.2L)\cos.L}{2905.5}. \quad . \quad . \quad . \quad 10$$

The vertical action of this centrifugal force will be as the *cosine* of the latitude.

The deviation v of a plumb-line from the vertical, caused by centrifugal force of the earth's rotation, will be

$$\tan.v=\frac{F\sin.L}{W}. \quad . \quad . \quad . \quad . \quad . \quad 11$$

The deviation of the surface of a liquid from the true horizon can be calculated from the same formulas.

§ 158. CENTRIFUGAL FORCE OF A BODY MOVING ON THE SURFACE OF THE EARTH.

A body moving on the surface of the earth loses in weight the faster it moves, and it may move so fast that it loses all its weight, which happens when the centrifugal force is equal to the weight or force of attraction.

Centrifugal force, $F=\dfrac{W v^2}{g\ R}$, in which

$W=$ weight of the moving body in pounds.
$v=$ velocity of the body in feet per second.
$R=$ radius of the earth in feet.

Example 3. A railway train weighing 1000 tons or $W = 2240000$ pounds is running on a horizontal track at the rate of 60 miles per hour, or velocity $v = 88$ feet per second.

Required the centrifugal force of the train in the curvature of the earth?

$$F = \frac{W v^2}{g R} = \frac{2240000 \times 88^2}{32.17 \times 20887680} = 25.036 \text{ pounds.}$$

The train of 1000 tons lost only 25 pounds by the centrifugal force.

This centrifugal force is without regard to the earth's rotation around its axis.

The centrifugal force of a body moving on the surface of the earth in regard to the earth's radius depends upon the direction of motion. When moving from east to west there is less centrifugal than when moving from west to east.

The centrifugal force of a body at rest or in motion on the earth's surface affects the weight of that body when weighed on a spring balance.

When the centrifugal force of the moving body is equal to the weight, or $F = W$, then $g R = v^2$, of which the velocity $v = \sqrt{g R}$.

The radius of the earth is about 3956 miles or 20887680 feet, which inserted in the formula will give a velocity

$$v = \sqrt{32.17 \times 20887680} = 29528 \text{ feet, or } 5.5925 \text{ miles per second.}$$

That is to say, a body moving with a velocity of 5.6 miles per second at or near the surface of the earth would not fall to the earth, because its centrifugal force is equal to its weight.

A cannon-ball fired horizontally from the top of a high mountain, and with a velocity of 5.6 miles per second, would continue to rotate around the earth and make one revolution in about seven minutes and a half.

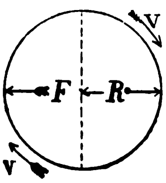
This is the principle upon which the moon rotates around the earth—namely, that the centrifugal force of the moon is equal to the centripetal force or force of attraction between the two bodies.

The moon, however, does not revolve around the earth's centre, but around the common centre of gravity of the two bodies, which is at about 0.7 of the earth's radius from the centre of the earth.

§ 159. CENTRIFUGAL FORCE OF A ROUND DISK ROTATING AROUND ITS CENTRE.

The dotted line represents any diameter dividing the disk into two equal parts. When rotating each part tends to separate in that diameter with a force equal to the centrifugal force of each half. The radius of the centrifugal force is the distance from the centre of rotation to the centre of gravity of each half. The centre of gravity of a semicircular plane is 0.424 of the radius from the centre.

Fig. 185.



W = weight in pounds of half the disk, which may be of any thickness, say a cylinder.
R = outside radius in feet of the disk or cylinder.
n = revolutions per minute.

Centrifugal force, $$F = \frac{W(0.424\,R)n^2}{2933.5} \quad . \quad . \quad . \quad 1$$

Revolution, $$n = \sqrt{\frac{2933.5\,F}{W(0.424\,R)}} \quad . \quad . \quad . \quad 2$$

Example 1. A grinding-stone of 6 feet in diameter and 1 foot thick is making $n = 100$ revolutions per minute. Required the centrifugal force?

The specific gravity of sandstone is about 2.5, when the weight of the grinding-stone will be

$$6^2 \times 0.785 \times 62.33 \times 2.5 = 4405.8 \text{ pounds.} \qquad W = 2202.9.$$

Radius of grindstone 3 feet, and $3 \times 0.424 = 1.272$ feet, the centrifugal radius.

Centrifugal force, $$F = \frac{2202.9 \times 1.272 \times 100^2}{2933.5} = 9553.6 \text{ pounds.}$$

Example 2. The tensile strength of ordinary sandstone may be limited to 100 pounds per square inch of section. Required the number of revolutions per minute of the grinding-stone in the preceding example, at which the centrifugal force would break it?

The section in the diameter of the stone is $72 \times 12 = 864$ square inches, which multiplied by the tensile strength 100 will be $F = 86400$.

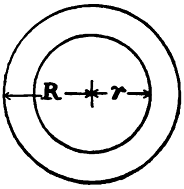
Revolution, $$n = \sqrt{\frac{2933.5 \times 86400}{2202.9 \times 1.27}} = 301 \text{ per minute.}$$

§ 160. CENTRIFUGAL FORCE OF A RING.

The centrifugal of a ring of square or rectangular section is

$$F = \frac{W n^2 \sqrt{R^2 + r^2}}{2933.5}. \quad . \quad . \quad 1$$

Fig. 186.



W = half the weight of the ring.

$$n = \sqrt{\frac{2933.5\,F}{W\sqrt{R^2+r^2}}}. \quad . \quad . \quad 2$$

Example 1. The ring of a cast-iron fly-wheel is 9 inches square, the outer radius $R = 6$ feet, and $r = 5.25$, revolutions per minute $n = 64$. Required the centrifugal force of the fly-wheel?

The weight of the ring will be

$$(113 - 86.59)0.75 \times 450 = 8910, \qquad W = 4455 \text{ pounds.}$$

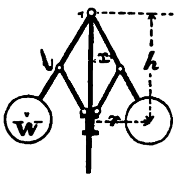
$$F = \frac{4455 \times 64^2\sqrt{6^2 + 5.25^2}}{2933.5} = 49.592 \text{ pounds.}$$

Example 2. The tensile strength of cast-iron is about 18000 pounds to the square inch, which multiplied by 81 square inches, the section of the ring, will be $1458000 \times 2 = 2916000$ pounds, the force F. Required the number of revolutions of the fly-wheel in the preceding example at which it would break by the centrifugal force?

$$\text{Revolution} \quad n = \sqrt{\frac{2933.5 \times 2916000}{4455\sqrt{6^2 + 5.25^2}}} = 490 \text{ per minute.}$$

§ 161. THE CENTRIFUGAL GOVERNOR.

Fig. 187.



The action of a governor is that the weight of each ball is to the centrifugal force as the height h is to the radius of rotation r.

$$W : F = h : r.$$

$$W = \frac{Fh}{r}, \qquad F = \frac{Wr}{h}, \qquad h = \frac{Wr}{F}, \qquad r = \frac{Fh}{W}.$$

The angle x which the arms form with the centre-shaft is

$$tan.x = \frac{r}{h} = \frac{F}{W}, \qquad sin.x = \frac{r}{l}.$$

Centrifugal force $F = \frac{W r n^2}{2933.5} = W\ tan.x$,

of which $n = \sqrt{\frac{2933.5\ tan.x}{r}} = 54.16\sqrt{\frac{tan.x}{r}}$. . 1

$n = \frac{54.16}{\sqrt{l\ cos.x}} = \frac{54.16}{\sqrt{h}}$. . . . 2

These formulas give the number of revolutions per minute of the governor when the angle x, height h, radius r or length of the pendulum are given.

The weight of the balls does not influence the angle of the governor, because the centrifugal force varies as the weight. The weights only serve to do the work of regulating the steam-valve when the velocity of rotation changes.

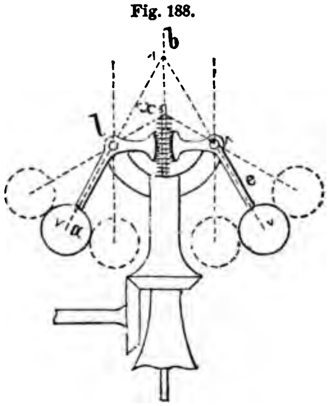
Example. The pendulum arms of a governor are $l = 2$ feet from the upper joint to the centre of the balls. How many revolutions n? must the governor make per minute to form an angle $x = 45°$?

$$n = \frac{54.16}{\sqrt{2 \times cos.45°}} = 46 \text{ revolutions.}$$

§ 162. THE VARIABLE PENDULUM GOVERNOR.

In this governor the balls are hung at a distance from the centre-line, by which the length of the pendulum varies with the angle x. The distance from the centre a of the ball to where the direction of the arm cuts the centre-line at b is the real length of the pendulum in that position of the governor. When the balls hang vertical, the length of the pendulum is infinite.

Fig. 188.



d = right-angular distance from point of suspension to centre-line in feet.

e = distance from centre of ball to point of suspension.

Then the length of the pendulum in any position of the balls will be in feet, $l = d\ cosec.x + e$.

The formulas and table for the ordinary centrifugal governor will also answer for this governor, only that the pendulum length must be measured from a to b.

Revolutions per minute of governors with different angles and lengths of the pendulum-arms.

Length	ANGLE OF PENDULUM IN DEGREES.								
l	20°	25°	30°	35°	40°	45°	50°	55°	60°
inches.	n	n	n	n	n	n	n	n	n
1	193	197	202	207	214	223	234	248	265
2	137	140	143	147	152	158	166	175	187
3	112	114	116	120	124	129	135	143	153
4	97	99	102	104	108	112	117	124	133
5	87	88	90	93	96	100	105	111	119
6	79	81	83	85	88	91	96	101	109
7	73	75	76	78	81	84	89	94	100
8	68	70	71	73	76	79	83	88	94
9	64	66	67	69	71	74	78	82	88
10	61	62	64	66	68	70	74	78	84
11	58	59	61	63	65	67	70	75	80
12	56	57	58	60	62	64	67	72	77
15	50	51	52	53	55	58	60	64	69
18	45	46	47	49	51	53	55	58	63
21	42	43	44	45	47	49	51	54	58
24	39	40	41	42	44	46	48	50	54
27	37	38	39	40	41	43	45	48	51
30	35	36	37	38	39	41	43	45	48
33	34	35	36	37	38	39	41	43	46
36	32	33	34	35	36	37	39	41	44
39	30	31	32	33	34	35	37	39	41
42	30	31	32	33	34	35	36	38	41
45	29	30	31	32	33	34	35	37	40
48	28	29	30	31	32	33	34	36	39

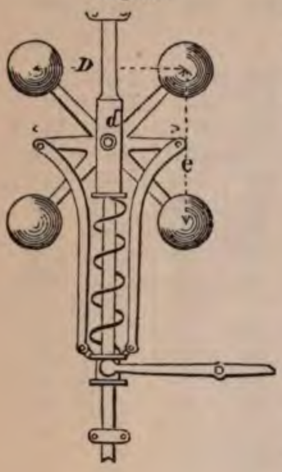
Vertical height *h* in inches of the centre of suspension above the centre of the balls, of governors making *n* revolutions per minute.

n	h	n	h	n	h	n	h
20	88.00	32	34.38	51	13.53	114	2.71
21	79.82	33	32.32	54	12.07	120	2.44
22	72.73	34	30.45	57	10.83	126	2.22
23	66.54	35	28.74	60	9.78	132	2.02
24	61.07	36	27.16	66	8.08	138	1.85
25	56.32	37	25.71	72	6.79	144	1.69
26	52.07	38	24.38	78	5.78	150	1.56
27	48.29	39	23.14	84	4.54	162	1.34
28	44.90	40	22.00	90	4.34	174	1.16
29	41.86	42	18.18	96	3.82	186	1.01
30	39.11	45	17.38	102	3.38	198	0.88
31	36.63	48	15.27	108	3.02	200	0.87

§ 163. ISOCHRONOUS GOVERNOR (Devised by the Author).

The construction of this isochronous governor is readily understood by the illustration. It is perfectly balanced, and will work equally well in any position (horizontal, inclined, or vertical) it may be placed. It is independent of the force of gravity.

Fig. 189.



W = weight in pounds of the four balls.

F = centrifugal force.

f = force in pounds on the spring in the direction of the spindle.

D = diameter in feet of the circle described by the centre of the balls.

d = length of the two levers, as shown on the drawing.

e = distance between the balls in the direction of the spindle.

The centrifugal force of the four balls will be

$$F = \frac{W D n^2}{5867}.$$

$$F : f = d : e, \quad \text{of which} \quad f = \frac{F e}{d}.$$

$$f = \frac{W D e n^2}{5867\, d}, \quad \text{and} \quad n = \sqrt{\frac{5867 f d}{W D e}}.$$

The force f of the spring ought to be so adjusted that the balls will form a square when the governor runs at the average speed.

§ 164. CENTRIFUGAL FORCE OF BODIES MOVING ON CURVED ROADS.

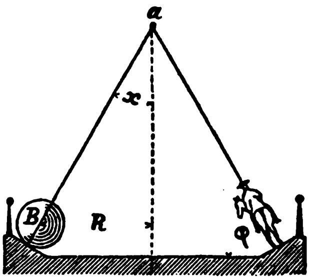
Fig. 190 represents a section of a circus ring, and a rider on a horse. It is well known that the faster the horse runs the more he leans toward the centre of the ring.

Let the body B be suspended from a by the line $B\,a$, and swung around the circle in the same path and with the same velocity as that of the horse; then the inclination of the line $B\,a$ will be the same as that of the horse and the rider. From the formulas for the centrifugal governor we have the angle x as follows:

$$\tan.x = \frac{R\, n^2}{2933.5}. \quad . \quad . \quad . \quad . \quad 1$$

R = radius in feet of the centre of the track.
n = number of turns around the circle per minute.
The rise of the track above horizon will be the same as the angle x.
Let V denote the velocity in feet per second of the horse or ball.

Fig. 190.



$$V = \frac{2\pi R n}{60}. \quad . \quad . \quad 2$$

$$n = \frac{60\,V}{2\pi R}. \quad . \quad . \quad 3$$

$$tan.x = \frac{R\,60^2\,V^2}{2933.5(2\pi R)^2}. \quad 4$$

$$tan.x = \frac{3600\,V^2}{2933.5 \times 4 \times 9.86955\,R} = \frac{V^2}{32.17\,R}. \quad . \quad . \quad 5$$

This formula gives the angle x when the radius R and velocity V are given. When we have the velocity expressed in statute miles per hour, the angle x will be: $tan.x = \frac{\text{Miles}^2}{14.956\,R}$. 6

In railway curves the outer rail should be elevated an angle x above the inner rail. Call G = width of gauge in inches. h = elevation of the outer rail in inches.

$$\frac{h}{G} = \frac{\text{Miles}^2}{14.956\,R}. \quad 7 \text{ and } 8. \quad h = \frac{G\,\text{Miles}^2}{15\,R}.$$

This formula is not strictly correct, because $\frac{h}{G}$ is *sin.x*, instead of *tan.x*, but the difference is of no practical importance in the small angle of elevation of the outer rail in railroad curves.

Example. A railroad train is to run at the rate of 30 miles per hour on a curve of $R = 500$ feet radius, anj gauge $G = 56.5$ inches. d
Required the elevation of the outer rail?

$$h = \frac{56.5 \times 30^2}{15 \times 500} = 6.8 \text{ inches.}$$

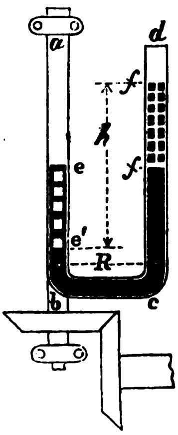
With this elevation the wheels would bear equally on both rails. For slower speed on the same curve the wheels would bear heavier on the inner rail than on the outer one.

§ 165. TO MEASURE ANGULAR VELOCITY BY CENTRIFUGAL FORCE.

The illustration, Fig. 191, represents a glass tube bent into the shape of a fork a, b, c, d, and filled with mercury to the line $e\,f$.

The leg $c\,d$ is set into rotation around the other leg $a\,b$ as axis, and the centrifugal force of the mercury in the part $b\,c$ will raise the mercury in the leg $c\,d$ and lower it in $a\,b$ to a difference h of level in the two legs.

Fig. 191.



A = area of the cross-section of the tube in square inches.

h = difference of height of the mercury in inches.

R = radius of revolution of the leg $c\,d$ in inches.

n = number of revolutions per minute.

W = weight in pounds of the column of mercury of the height h.

w = weight of the mercury in the part $b\,c$.

The weight of a cubic inch of mercury at the temperature of 60° Fahr. is 0.941 of a pound.

The weights, $W = 0.941\,A\,h$, and $w = 0.941\,A\,R$.

The centrifugal force of the mercury w will be

$$F = \frac{4\,w\,R\,\pi^2\,n^2}{2 \times 12 \times 60^2 \times g} = \frac{w\,R\,n^2}{70405},$$

which must be equal to the weight W which acts as centrifugal force. Insert the values of W and w in the formulas, and we have

$$0.941\,A\,h = \frac{0.941\,A\,R^2 n^2}{70405}.$$

$$h = \frac{R^2\,n^2}{70405}, \qquad R = \frac{241.44}{n}\sqrt{h}, \qquad n = \frac{241.44}{R}\sqrt{h}.$$

The differential height h of the mercury is independent of the sectional area A, which can be irregular.

The parts $a\,b$ and $c\,d$ of the tube should be parallel, but the part $b\,c$ need not be at right angles to the legs, nor need it be straight, but can be made of any curve or shape.

Example 1. The radius between the centre lines of the legs is $R = 3$ inches, and makes $n = 184$ revolutions per minute. Required the differential height h?

$h = \dfrac{3^2 \times 84^2}{70405} = 4.327$ inches, of which one-half, or 2.1635 inches,

will fall in the leg $a\,b$ and rise the other half in the leg $c\,d$.

Example 2. How many revolutions must the instrument make with a radius $R=5$ inches, to raise a differential column of $h=16$ inches?

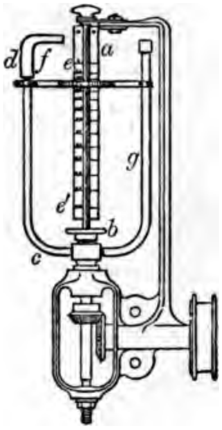
$$n=\frac{241.44}{5}\sqrt{16}=193.152 \text{ revolutions per minute.}$$

When the areas of the cross-sections of the bore in the two legs are alike, the column of mercury will sink in the centre leg $a\ b$ as much as it rises in the leg $c\ d$, or $e\ e'=f\ f'$, and a graduated scale could be attached to the centre leg to indicate the number of revolutions of the instrument. It is not necessary to make the cross-sections of the legs alike, as will be explained in the following section.

§ 166. REVOLUTION INDICATOR.

This instrument, Fig. 192, is based upon the principles described in the preceding section, and invented and patented by Edward Brown of Philadelphia.

Fig. 192.



The centre tube $a\ b$ is made of glass and contains the mercury which communicates with the iron tube $c\ d$. When the indicator stands still the mercury level is at $f\ e$.

The iron tube is fitted with a vessel at d, sufficiently large to contain the whole column $e\ e'$ of mercury, so that when the indicator revolves with its highest speed, the mercury will fall from e to e' in the glass tube, and rise to a small height in the vessel d. The part g is only a balance rod of solid iron.

The vessel d is turned inward at the top to prevent the mercury from splashing out.

The graduated scale is held in position by the framing, as shown by the illustration.

The instrument is intended for indicating the rate revolutions of a steam-engine or other rotary machine, for which purpose it is not necessary that the indicator should make the same number of revolutions, but can be geared to run with any desired speed, and the scale is graduated to suit the gearing and indicate directly the number of revolutions of the engine.

A = area of cross-section of the vessel d, and a = that of glass tube.

h = difference of height of the mercury.

x = depth to which the mercury sinks in the glass tube in inches.

n = revolutions per minute of the indicator.

R = radius of rotation of the iron tube $c\ d$.

$$h=\frac{R^2 n^2}{70405},\qquad h=x\left(1+\frac{a}{A}\right),\qquad x=\frac{R^2 n^2}{70405\left(1+\frac{a}{A}\right)}.$$

Example. The areas $a : A = 1 : 16$, and $R = 4$ inches, making $n = 15$ revolutions per minute. Required the sinkage x of mercury in the glass tube?

$$x=\frac{4^2\times 150^2}{70405(1+\frac{1}{16})}=4.8125 \text{ inches.}$$

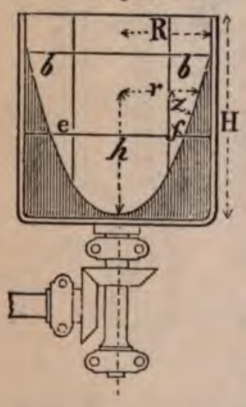
The indicator is geared, say 3 to 1 revolutions of the engine; the latter will make 50 turns per minute when the mercury sinks 4.8125 inches in the glass tube. The divisions on the scale will be a geometrical progression, or as the square of the revolutions.

The scale can thus be graduated, but number the divisions so as to correspond with the revolutions of the engine.

§ 167. CENTRIFUGAL FORCE OF A LIQUID ENCLOSED IN A ROTATING VESSEL.

Fig. 193 represents a cylindrical vessel of radius R and height H, about half filled with any liquid, say to the line $e\,f$. When the system is set into rotation the centrifugal force will form the liquid into a complement paraboloid, or the section of the liquid will be a parabola, in accordance with the following formula:

Fig. 193.



$$h=\frac{r^2 n^2}{70405},\quad r=\frac{241.44}{n}\sqrt{h},\quad n=\frac{241.44}{r}\sqrt{h}.$$

h = abscissa and r = ordinate.

The formulas give the form of the parabola in the rotating vessel.

The ribs $b\ b$ are fastened inside to make the liquid rotate with the vessel.

The angle z at any point of the parabola with the ordinate radius r will be $tan.z=\frac{2\,h}{r}$.

The same angle will be formed by the surface of the mercury in the vessel d, Fig. 192.

The cubic content of a paraboloid is one-half of that of a cylinder of the same base and height; the liquid in a rotating cylindrical vessel (Fig. 193) will therefore sink in the centre as much as it rides on the sides until the inverted vertex of the parabola reaches the bottom of the vessel.

§ 168. PENDULUM.

A body freely suspended above its centre of gravity and made to swing is called a pendulum.

Simple Pendulum is the combination of a small body hung on a light line and made to swing.

Compound Pendulum is a rigid system of bodies suspended above the centre of gravity and free to swing.

A body suspended at c by the line l will hang perpendicular under the point of suspension, and the combination is called a *plumb line*. Draw the body aside from b to a, and leave it free to the action of gravity, which will draw the body back to b. The primitive work consumed in drawing the body aside is stored into it by the force of gravity which moves the body from a to b, where it arrives with a velocity equal to that due to an equal height of fall S. A body in motion cannot be stopped without restoring the work which has set it in motion, for which resistance is required, and as the body met with no resistance at b, its motion will be continued to c. Whilst moving from b to c the body meets the resistance of gravity, which has discharged the work when arriving at c, an equal height S above b. As the body is free to move in the arc of the circle, the force of gravity will draw it back to b, and the motion continue to a, where again it will be drawn back to b, and so it will continue to move fore and back for ever if no other force interferes with the operation. This operation is called *oscillation of a pendulum.*

Fig. 194.

§ 169. DYNAMICS OF THE PENDULUM.

Let a body M, Fig. 195, be suspended by an inflexible rod without weight from the centre C, and free to revolve in the whole circle around that centre.

R = radius of the circle in feet.

φ = angle of the rod with the vertical at any position of the body M in the circle.

The body is so supported in its centre of gravity that when it swings it will not rotate with the rod; that is to say, the axis a, b will remain vertical in any position of the body in the circle. Then each

particle of matter in the body will describe equal circles of radii R. Move the body M to the highest position on the circle at O, where it will be balanced on the rod which takes up the whole weight W, so that the body will remain stationary; but the slightest force tending to move the body toward 1 or 15 will disturb the equilibrium, and the force of gravity with the co-operation of the rod will cause it to describe a circle. Suppose the motion to be from O toward 1, and let the lines W represent the weight of the body, which is the force of gravity. In the several positions 1, 2, 3, 4, etc. of the body, resolve the weight W into two forces, one acting in the direction of motion or the tangent, and one parallel with the radius of that position. The body is then moved only by the force F acting in the direction of the tangent, and the other force acting parallel with the radius, or at right angles to the direction of motion, has no effect upon the body's motion.

$$F = W \sin.\varphi.$$

Fig. 195.

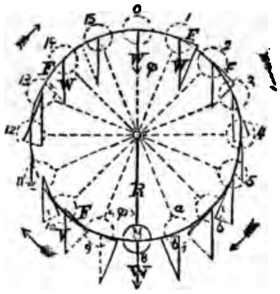
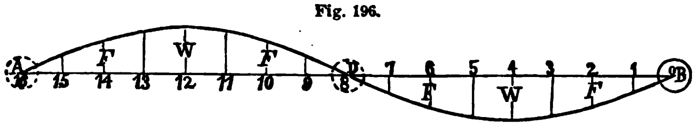


Fig. 196.



Draw a straight line A, B, Fig. 196, equal to the length of the circumference of the circle, and divide it into an equal number of parts as the division of the circle. At each division set off the corresponding force F at right angles to A, B, and join these forces with the curve $B\,b\,D\,c\,A$. Now, if the body M be placed at B and the same forces F acting upon it in the direction B, A, it will attain the same velocity as that at the corresponding divisions in the circle.

The body will attain its greatest velocity at D, which corresponds with that at the lowest position in the circle; after which the forces F act opposite the motion and stop it at A, which corresponds with

the highest position O in the circle. The work accomplished by the forces F is represented by the area of the figure bounded within the curve $B\,b\,D\,c\,A$ and the straight line A, B.

The area $B\,b\,D$ represents the work of drawing the body from O via 1, 2, 3, etc. to 8, and which is equal to the weight W multiplied by the diameter of the circle, or

$$K = W\,2\,R = \frac{M\,V^2}{2}, \quad . \quad . \quad . \quad . \quad 1$$

in which V = velocity of the mass M when passing the lowest position at the division 8.

This work is stored in the body, and cannot be taken out of it without realizing an equal amount of work, which is accomplished by the body continuing its motion in the circle against the action of the force of gravity, until it reaches its starting position at the highest point O.

The work accomplished by the force of gravity at any position of the mass M is the weight W multiplied by the vertical space, or *sinus-versus* of the angle φ moved through—namely,

$$K = W\,R\,sinv.\varphi = \frac{M\,V^2}{2}. \quad . \quad . \quad . \quad 2$$

The velocity with which the body passes any division or point in the circle will be

$$V = \sqrt{\frac{2\,W\,R\,sinv.\varphi}{M}}. \quad . \quad . \quad . \quad . \quad 3$$

$M = \dfrac{W}{g}$, which inserted in Formula 3 will be

$$V = \sqrt{2\,g\,R\,sinv.\varphi}. \quad . \quad . \quad . \quad . \quad 4$$

When the body has moved from the highest to the lowest position it has described an angle $\varphi = 180°$, for which $sinv. = 2$ and the velocity

$$V = \sqrt{4\,g\,R}. \quad . \quad . \quad . \quad . \quad . \quad 5$$

§ 170. TIME OF OSCILLATION.

When the body passes the lowest position, say between the 7th and 9th divisions, the force curve is nearly a straight line, and we can therefore assume the motive force to be in proportion to the length of the arc from the lowest or 8th division to the position of the body. Then when the body is oscillating a small angle, the motive force is directly as the space from the lowest position, which corre-

sponds with the case in § 92, in which it is proved that the time is independent of the space of action, or that the body will oscillate equal times independent of the angle of oscillation.

From Formula 3, § 92, we have the time of half an oscillation to be

$$T=\sqrt{\frac{M}{C}}. \quad . \quad . \quad . \quad . \quad . \quad 1$$

For a full single oscillation, $T=2\sqrt{\frac{M}{C}}$. 2

It is assumed in this formula that the motive force is a function of the arc, or $F=C\,S$.

$S=\frac{\pi R \varphi}{180}$, and $F=\frac{C \pi R \varphi}{180}$, of which the constant $C=\frac{180\,F}{\pi R \varphi}$. 3

The force F for one oscillation will be

$$F=g\,M.\frac{\pi R \varphi}{180}.\frac{4}{R\,\pi^2}=g\,M\frac{4\,\varphi}{180\,\pi}. \quad . \quad . \quad . \quad 4$$

Insert this value of F in Formula 3—namely,

$$C=\frac{180\times 4\,\varphi\,g\,M}{\pi\,R\,\varphi\,180\,\pi}=\frac{4\,g\,M}{\pi^2\,R}. \quad . \quad . \quad . \quad . \quad 5$$

Insert this value of the constant C in Formula 2, and the time of one single oscillation will be

$$T=2\sqrt{\frac{M\pi^2 R}{4\,g\,M}}=\pi\sqrt{\frac{R}{g}}. \quad . \quad . \quad . \quad . \quad 6$$

The length R of a pendulum oscillating T seconds will then be

$$R=\frac{g\,T^2}{\pi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 7$$

Example. Required the length of a pendulum making one single oscillation per second?

$$R=\frac{32.17\times 1^2}{3.1416^2}=3.259353 \text{ feet.}$$

§ 171. ELEMENTS OF THE PENDULUM.

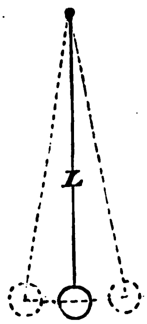
L = length in inches of pendulum from centre of suspension to centre of oscillation. In the simple pendulum the centre of oscillation is in the centre of gyration.

n = number of oscillations in T seconds.

T = time in seconds in which the pendulum makes n single oscillations.

Simple Pendulum.

Fig. 197.



$$L = \frac{12\,g\,T^2}{\pi^2\,n^2} = \frac{39.114\,T^2}{n^2}.$$

$$T = \sqrt{\frac{L\,n^2}{39.114}} = \frac{n\sqrt{L}}{6.2541}.$$

$$n = \sqrt{\frac{12\,g\,T^2}{\pi^2\,L}} = \frac{6.2541\,T}{\sqrt{L}}.$$

$$g = \frac{L\,\pi^2\,n^2}{12\,T^2} = 0.822467\frac{L\,n^2}{T^2}.$$

If the suspended weight is spherical of radius r and l = length between centre of suspension and centre of ball, the pendulum length L will be $L = \sqrt{l^2 + \frac{2}{5}r^2}$ in inches.

Delicate pendulum experiments must be made in a vacuum in order to agree with the formulas, and the acceleratrix g must be calculated for the latitude and elevation above the level of the sea where the experiment is made.

The length of a pendulum in feet for the oscillation of seconds, is equal to the acceleratrix g, divided by π^2.

The following table gives the pendulum length in inches, oscillating seconds with the corresponding value of the acceleratrix g in various places:

Locations.	Pendulum, L.	Generatrix, g.	Latitude. $D.$ $m.$ $s.$
At the Equator........................	39.0152	32.0789	0° 0′ 0″
At Washington, D. C..............	39.0958	32.1552	38 53 23
At New York..........................	39.1017	32.1608	40 42 40
At Mean Radius of the Earth...	39.1270	32.1808	45 00 00
At London..............................	39.1398	32.1912	51 31
At Stockholm..........................	39.1845	32.2281	59 21 30
At Spitzbergen........................	31.2147	32.2528	79

The length of the pendulum for seconds, at the level of the sea in any latitude, will be

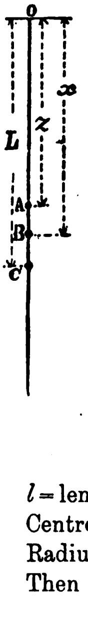
$$L = 39.127 - 0.09982 \cos.2 \text{ latitude}.$$

When the latitude is over 45° the cosine for double the latitude will be negative, which multiplied by the negative coefficient -0.09982 gives a positive product to be added to the pendulum length 39.127 of 45° latitude.

§ 172. ELEMENTS OF THE COMPOUND PENDULUM.

Fig. 198 represents a rod suspended at o, and free to swing.

Fig. 198.



z = distance from the centre of suspension o to the centre of gravity A. As the rod is parallel, its centre of gravity A will be in the middle.

x = distance from centre of suspension to centre of gyration B, which is the radius of gyration.

L = distance from the centre of suspension to the centre of oscillation, which is the pendulum length of the rod.

These distances bear the following relation to one another:

$$z : x = x : L.$$

That is to say, the radius of gyration x is the mean proportion between z and L. This proportion will hold good for any kind of compound pendulum.

$$z = \frac{x^2}{L}, \qquad x = \sqrt{z\,L}, \qquad L = \frac{x^2}{z}.$$

l = length of the whole rod.
Centre of gravity $z = \frac{1}{2}\, l$.
Radius of gyration $x = 0.57735\, l$.
Then the pendulum length of a rod or bar will be

$$L = \frac{x^2}{z} = \frac{(0.57735\, l)^2}{0.5\, l} = 0.6666\, l,$$

or the pendulum length is $\frac{2}{3}$ of the whole length of the bar.

In a compound pendulum the particles of matter located near to the centre of suspension have a tendency to oscillate faster, and those at the greatest distance tend to oscillate slower, than does the compound pendulum; and as all the particles are rigid into one body, there must be one of them which has no tendency to oscillate faster or slower. This particle is called the centre of oscillation, and if it

was suspended alone as a single pendulum, it would oscillate the same time as does the compound pendulum.

Professor Huyghens of Holland discovered that the centres of suspension and oscillation are convertible into one another; which is to say that if the compound pendulum be suspended in its centre of oscillation, the former centre of suspension will then be the centre of oscillation, and it will oscillate the same lengths of time.

The pendulum length of a compound pendulum can thus be ascertained with great precision by suspending it in two different points alternately, so that it will oscillate equal lengths of time in both cases.

The experiments of pendulum oscillation should be made under vacuum, in order to avoid the influence of the resistance of air.

The value of the acceleratrix g can be determined with great precision by a pendulum whose length is accurately known, for which purpose a compound pendulum in form of a parallel rod or bar appears to be the best.

$$g = \frac{L\,\pi^2\,n^2}{12\,T^2}.$$

§ 173. RADIUS OF GYRATION.

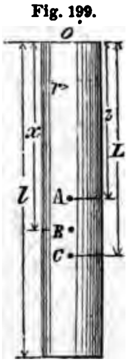
The radius of gyration of an oscillating body can be accurately determined by the number of oscillations n in the time T.

$$L = \frac{12\,g\,T^2}{\pi^2\,n^2} = \frac{x^2}{z}.$$

$$\text{Radius gyration} \quad x = \sqrt{\frac{12\,g\,T^2 z}{\pi^2\,n^2}} = \frac{6.2541\,T\sqrt{z}}{n}.$$

§ 174. TO FIND THE PENDULUM LENGTH OF A CYLINDER SUSPENDED AT ONE END.

Fig. 199.



l = length, and r = radius of cylinder.

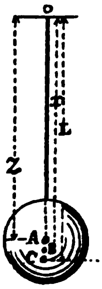
$$\text{Centre of gravity} \quad z = 0.5\,l.$$

$$\text{Radius of gyration} \quad x = \sqrt{\frac{4\,l^2 + 3\,r^2}{12}}.$$

$$\text{Pendulum} \quad L = \frac{x^2}{z} = \frac{4\,l^2 + 3\,r^2}{6\,l}.$$

§ 175. PENDULUM LENGTH OF A SUSPENDED BALL.

Fig. 200.



z = distance from centre of suspension to centre of gravity of the ball.

r = radius of the ball.

Radius of gyration $x = \sqrt{z^2 + \frac{2}{5}r^2}$.

Pendulum $L = \dfrac{x^2}{z} = \dfrac{z^2 + 0.4\,r^2}{z}$.

§ 176. COMPOUND PENDULUM OF TWO RIGID BALLS.

Fig. 201.

R and r = radii of the balls.

a and b = distances from centre of suspension to centres of gravity of the balls.

Centre of gravity $z = \dfrac{P\,a + Q\,b}{P + Q}$.

Radii of gyration, $\begin{cases} x' = \sqrt{a^2 + 0.4\,R^2}. \\ x'' = \sqrt{b^2 + 0.4\,r^2}. \end{cases}$

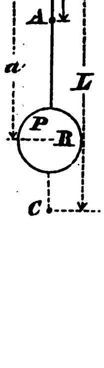
$$x^2(P+Q) = P\,x'^2 + Q\,x''^2, \quad x = \sqrt{\frac{P\,x'^2 + Q\,x''^2}{P+Q}}.$$

$$x = \sqrt{\frac{P(a^2 + 0.4\,R^2) + Q(b^2 + 0.4\,r^2)}{P+Q}}.$$

Pendulum $L = \dfrac{x^2}{z} = \dfrac{P(a^2 + 0.4\,R^2) + Q(b^2 + 0.4\,r^2)}{P\,a + Q\,b}$.

§ 177. DOUBLE COMPOUND PENDULUM.

Fig. 202.



Notation of letters are the same as in the preceding paragraphs.

Centre of gravity $z = \dfrac{P\,a - Q\,b}{P + Q}$.

Radii of gyration, $\begin{cases} x' = \sqrt{a^2 + 0.4\,R^2}. \\ x'' = \sqrt{b^2 + 0.4\,r^2}. \end{cases}$

Pendulum $L = \dfrac{x^2}{z} = \dfrac{P(a^2 + 0.4\,R^2) + Q(b^2 + 0.4\,r^2)}{P\,a - Q\,b}$.

When the pendulum length is given, the oscillations are calculated by the formulas for the simple pendulum.

Example. Weight and dimensions of the double compound pendulum balls are as follows:

Weight $P=50$ pounds, radius $R=4$ inches, lever $a=25$ inches.

Weight $Q=20$ pounds, radius $r=3$ inches, lever $b=15$ inches.

Required the pendulum length L? and single oscillations per minute n?

$$L=\frac{50(25^2+0.4\times 4^2)+20(15^2+0.4\times 3^2)}{50\times 25-20\times 15}=37.845 \text{ inches.}$$

Oscillations per minute, § 171, will be

$$n=\frac{6.2541\times 60}{\sqrt{37.845}}=60.9975.$$

It is nearly a second pendulum.

The nearer the double compound pendulum is suspended from its centre of gravity, the longer will be the pendulum length.

Pendulum Lengths for Different Numbers of Vibrations per Minute, and Time of each Vibration in Seconds.

No.	Feet.	Time.	No.	Inches.	Time.	No.	Inches.	Time.
1	11734	60	28	179.60	2.15	80	22.001	0.750
2	2935.5	30	29	167.43	2.08	84	18.181	0.712
3	1303.7	20	30	156.45	2.000	90	17.384	0.666
4	733.37	15	32	137.51	1.88	96	15.267	0.625
5	462.36	12	34	121.84	1.77	102	13.534	0.586
6	325.85	10	36	108.65	1.67	108	12.072	0.555
7	239.47	8.8	38	97.512	1.58	114	10.835	0.525
8	183.35	7.50	40	88.005	1.50	120	9.7785	0.500
9	144.86	6.66	42	79.825	1.43	132	8.0815	0.452
10	117.34	6.00	44	72.731	1.37	144	6.7900	0.416
11	96.975	5.45	46	66.545	1.31	156	5.7860	0.383
12	81.462	5.00	48	61.071	1.25	168	4.5452	0.356
13	69.433	4.62	50	56.325	1.20	180	4.3460	0.333
14	59.867	4.30	52	52.075	1.16	192	3.8168	0.313
15	52.15	4.00	54	48.290	1.12	204	3.3835	0.295
16	45.838	3.76	56	44.900	1.08	216	3.0180	0.279
17	40.602	3.53	58	41.860	1.04	228	2.7087	0.265
18	36.215	3.34	60	39.114	1.000	240	2.4445	0.250
19	32.504	3.17	62	36.631	0.965	252	2.2173	0.237
20	29.335	3.00	64	34.379	0.934	264	2.0204	0.228
21	26.608	2.86	66	32.326	0.907	276	1.8485	0.218
22	24.244	2.73	68	30.450	0.882	288	1.6975	0.209
23	22.183	2.62	70	28.737	0.855	300	1.5646	0.200
24	20.365	2.50	72	27.162	0.833	324	1.3444	0.185
25	18.775	2.41	74	25.714	0.818	348	1.1627	0.173
26	17.368	2.32	76	24.378	0.790	372	1.0175	0.162
27	16.097	2.23	78	23.144	0.769	396	0.88105	0.152

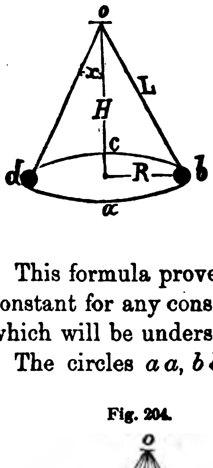
§ 178. CONICAL PENDULUM.

The swing of a pendulum can be made to describe a cone, the vertex of which is in the centre of suspension, and the base, which may be a circle or an ellipse, is at the lowest point of the pendulum. This is called a conical pendulum, and is illustrated by Fig. 203.

Let the body b be suspended from o and made to revolve in the circle a, b, c, d; then the height H may represent the weight W of the body, radius R the centrifugal force F, and L the strain f in the direction of the pendulum.

x = angle of the pendulum with the axis H.

Fig. 203.



Then, $F : W = R : H$.

$$F = \frac{WR}{H}, \quad W = \frac{FH}{R}, \quad R = \frac{FH}{W}, \quad H = \frac{WR}{F}.$$

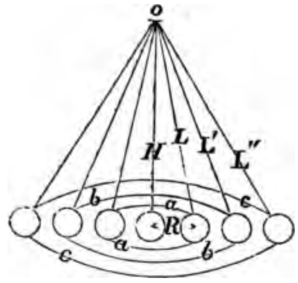
$$F = \frac{WRn^2}{2933.5} = \frac{WR}{H}, \quad \text{of which } n^2 = \frac{2933.5}{H}.$$

Revolutions per minute, $n = \sqrt{\dfrac{2933.5}{H}}$.

This formula proves that the number of revolutions per minute are constant for any constant height H, and independent of the angle x, which will be understood by Fig. 204.

The circles $a\,a$, $b\,b$ and $c\,c$ described by pendulums L, L', L'' are in the same plane, and H the height of the centre of suspension o above that plane. Any length of pendulum L, L', L'' describing a circle, a cone of any constant height H will make the same periodical revolutions or angular velocity.

Fig. 204.



Let t denote the time in seconds of one revolution of the conical pendulum.

Then, $t = \dfrac{60}{n}$, and $n = \dfrac{60}{t}$.

$$\frac{60}{t} = \sqrt{\frac{2933.5}{H}}, \quad \text{and } t = 1.1078\sqrt{H}.$$

The time of one revolution of a conical pendulum is directly as the square root of the height H.

The time of a double oscillation of a pendulum in a plane is

$$T = 0.3198\sqrt{L}.$$

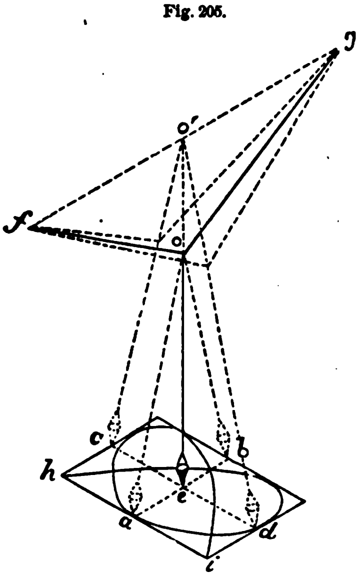
The time of one double oscillation of a pendulum in a plane is directly as the square root of the length L.

§ 179. THE OXFORD PENDULUM.

Fig. 205 represents a perspective view of a combination of two pendulums of different lengths invented at the Oxford University, England.

A line or wire f, o, g is fastened at f and g, at which point the line is free to swing. Another pendulum $o\ e$ is hung at o, so that $o\ f = o\ g$. The points of suspension f and o should be in a horizontal line f, o', g. A heavy body is hung on the pendulum at e to form the centre of oscillation. A rectangular screen a, c, b, d is placed with its centre under the pendulum, so that the centre line $a\ b$ is parallel with $f\ g$. The body e is a funnel filled with dry black sand, to fall on the screen and mark the course of the pendulum when set into vibration.

Fig. 205.



Place the funnel at a and let it vibrate the angle a, o, b, when the sand will mark the straight line $a\ b$ across the screen. Place the funnel at c and let it vibrate the angle c, o', d, when the sand will mark the straight line $c\ d$.

As the pendulum $o'\ e$ is longer than $o\ e$, it requires more time to vibrate from c to d than it does from a to b.

Now place the funnel at the corner h of the screen, and leave it to its own course of vibration. The short pendulum $o\ e$ will vibrate across the screen in periods of time equal to that when vibrating from a to b, whilst the long pendulum $o'\ e$ will vibrate the same periods as from c to d; the combined vibrations of the two pendulums will cause the funnel to describe a regular system of curves, which will be marked by the sand on the screen. The form of the figure so marked depends upon the proportion of lengths of the two pendulums.

Fig. 206.

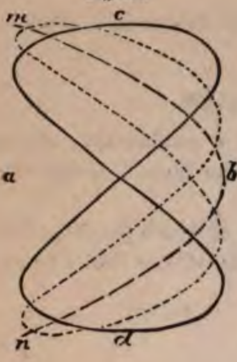


Figure 206. $L : l = 4 : 1$, $T : t = 2 : 1$.

The funnel is started from m with its natural course of vibration, and will trace the parabola $m\ b\ n$. On arriving at n the funnel will return in the same path viâ b to m, and repeat continually the same course shown by the dashed line.

If instead of allowing the funnel to take its own course from m, it is started by a lateral force in the direction of the dotted line, the funnel will describe that dotted line perpetually.

The funnel, started with proper velocity in any direction of the black line, would describe that figure 8 continually.

A variety of figures can thus be drawn with a constant proportion of pendulums, only by starting in different directions.

Fig. 207.

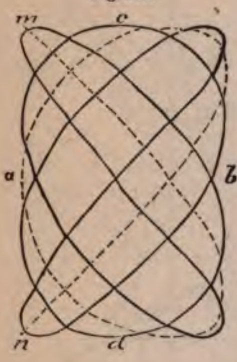


Figure 207. $L : l = 16 : 9$, $T : t = 4 : 3$.

The funnel started in its own course from m will describe the dotted line, and finally arrive at n, where it will return in the same path to m.

The dotted line is the figure drawn on the screen Fig. 205.

If the funnel is started with proper velocity in any direction of the full-drawn line it will describe that line perpetually.

It is almost impossible, or rather a chance, to start the vibration with such precision as to trace only that black line; but in practice the funnel will describe a more complicated and better-looking figure, which could not be constructed without extraordinary labor and patience.

The short pendulum will make 4 oscillations while the long one makes 3.

Fig. 208.

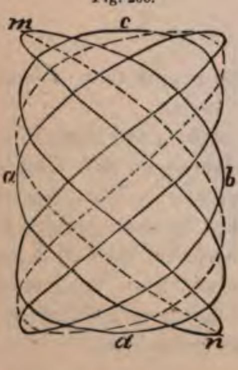


Figure 208. $L : l = 25 : 9$, $T : t = 5 : 3$.

This figure is described in the same way

as the preceding one, only with different proportion of pendulums. The long pendulum will make 3 oscillations while the shorter one makes 5.

Figure 209.

$L : l = 121 : 49, \quad T : t = 11 : 7.$

This figure is drawn only by the natural course of the pendulum starting from *m* and ending at *n*.

The short pendulum will make 11 oscillations whilst the long one makes 7.

Fig. 209.

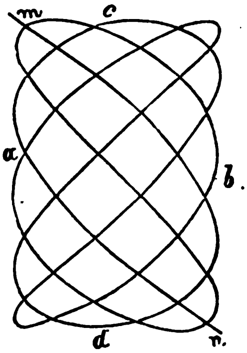


Figure 210.

$L : l = 529 : 169, \quad T : t = 23 : 13.$

The funnel is left to take its own course from the corner *m*, and traces the figure as shown by the illustration. When the funnel comes nearest to the corner *n'*, it will return in the direction of the dotted line and trace another figure between the lines, until finally it arrives at the corner *n*, when the whole figure is complete.

The short pendulum will make 23 oscillations whilst the long one makes 13.

Fig. 210.

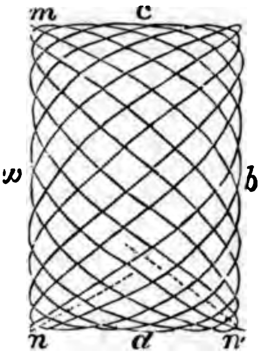


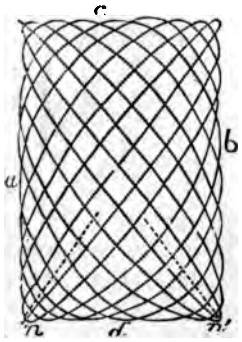
Figure 211.

$L : l = 529 : 361, \quad T : t = 23 : 19.$

This figure is similar to the preceding one, only that the short pendulum makes 23 oscillations whilst the long one makes 19.

It is supposed in the above figures that the angles of vibration of the two pendulums are constant, which is, however, not the case in practice. The resistance of the air to the funnel gradually diminishes the angles of vibration, so that the funnel can never return in its former course; which circumstance makes the figure more complicated and of a curious shading.

Fig. 211.



It is very difficult to represent the true appearance of the various figures by drawings.

When the angles of vibration are much reduced, the figure becomes darkest near the centre of the screen, and in the crossings the sand is piled up into regularly-formed heaps, which make a beautiful appearance.

An infinite variety of figures can thus be made by tracing one on the top of the other.

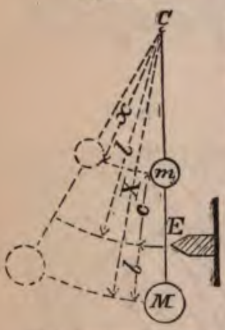
A machine could easily be constructed to make these figures perfectly, and which could be so arranged that a particular shading of a figure could not be reproduced even by the same machine.

§ 180. CENTRE OF PERCUSSION.

Take a bar of hard materials in your hands and strike it over a sharp edge, then if no shock is felt in the hands, the bar struck with its centre of percussion over the edge. If a downward shock is felt, the bar struck with its centre of percussion inside the edge, and with an upward shock outside the same. The work stored in the moving bar is discharged over the edge and in the hands. If the momentum in the moving bar is not evenly divided on both sides of the edge, the difference will be felt in the hands; but if evenly divided, all the work will be discharged on the edge and no shock felt in the hands.

Assume two bodies M and m hung on an inflexible line without weight and suspended at the point C, so as to form a pendulum. The edge E is placed at the centre of percussion of the bodies M and m, so that when the pendulum swings and strikes E, there will be no shock felt in the centre of suspension C. The condition under which this can be accomplished is that the momentums of M and m must be equally divided on both sides of E.

Fig. 212.



V = velocity of M, and v that of m, when the edge is struck. The momentums in the bodies will then be $M\,V$ and $m\,v$.

The letters X, l, x, b and c are as represented in Fig. 209.

When the bodies are moving around the common centre C, the velocities V and v will be as their radii X and x.

$$V : v = X : x.$$

The momentums of motion will then be $M X$ and $m x$, which multiplied by their respective levers of action b and c must be alike.

$$M X b = m x c.$$

$$b = X - l, \qquad \text{and} \quad c = l - x.$$

$$\text{Then} \qquad M X(X - l) = m x(l - x).$$

$$M X^2 - M X l = m x l - m x^2.$$

$$l(M X + m x) = M X^2 + m x^2.$$

$$l = \frac{M X^2 + m x^2}{M X + m x}.$$

The masses M and m can in this formula be expressed either in masses, weights or volumes.

Example. The masses $M = 5$ and $m = 4$ pounds, the radii of gyration $X = 3$ and $x = 2$ feet. Required the length l from the centre of oscillation to the centre of percussion?

$$l = \frac{5 \times 3^2 + 4 \times 2^2}{5 \times 3 + 4 \times 2} = 2.65 \text{ feet, or } 31.8 \text{ inches.}$$

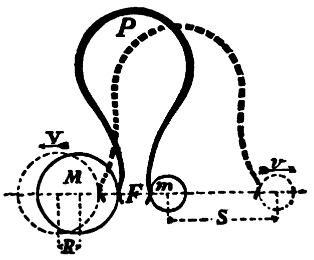
This length is the same as the pendulum length, or the centre of percussion is the same point as the centre of oscillation.

The centre of percussion is therefore calculated by the same formulas as for the pendulum or centre of oscillation.

A body or a system of bodies suspended in its centre of percussion will have its new centre of percussion in the former centre of suspension.

§ 181. ORDNANCE DYNAMICS.

Fig. 213.



Let a spring P of force F be applied between two masses M and m, free to move. As the forces F acting in opposite directions are alike, the spaces R and S and the velocities V and v will be inversely as the masses.

$$R : S = M : m, \qquad 1$$

$$V : v = M : m, \qquad 2$$

$$\text{and} \quad R : S = V : v. \qquad 3$$

If the force F is constant through the spaces R and S, the velocities will be

$$V = \sqrt{\frac{2FR}{M}}. \quad \ldots \quad 4$$

$$v = \sqrt{\frac{2FS}{m}}. \quad \ldots \quad 5$$

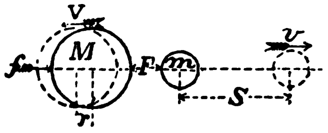
The work stored in each mass will be

$$K = FR = \frac{MV^2}{2}. \quad \ldots \quad 6$$

$$k = FS = \frac{mv^2}{2}. \quad \ldots \quad 7$$

In the case of firing a gun, M represents the mass of the gun and gun-carriage, m the mass of the ball and F the force of the gun-powder. The length of the bore of the gun passed through by the ball is $l = R+S$, and R is the recoil of the gun.

Fig. 214.



The recoil is generally partly counteracted by a force f applied on M as friction against the force F, as represented by Fig. 210.

$$\left.\begin{array}{r} (F-f) : M = V : T \\ F : m = v : T \end{array}\right\}. \quad T = \frac{MV}{(F-f)} = \frac{mv}{F}. \quad \ldots \quad 8$$

$$v = \frac{FMV}{m(F-f)}, \qquad V = \frac{mv(F-f)}{MF}. \quad \ldots \quad 9$$

$$V = \sqrt{\frac{2(F-f)r}{M}}, \text{ of which } r = \frac{MV}{2(F-f)}. \quad \ldots \quad 10$$

$$v = \sqrt{\frac{2FS}{m}}, \text{ of which } S = \frac{mv^2}{2F}. \quad \ldots \quad 11$$

$$l = S + r = \frac{mv^2}{2F} + \frac{MV^2}{2(F-f)}. \quad \ldots \quad 12$$

$$r = l - S = l - \frac{mv^2}{2F}.$$

$$\text{Recoil} \quad r = \frac{l}{\dfrac{FM}{m(F-f)} + 1} \quad \ldots \quad 14$$

This is the recoil at the moment the mass or ball m reaches the space S, but the mass M has then a velocity V, which must be stopped by the force f in an additional recoil r'.

$$r' = \frac{MV^2}{2f}, \text{ and the whole recoil } r+r' = \frac{l}{\frac{FM}{m(F-f)}+1} + \frac{MV^2}{2f}. \quad 15$$

This formula reduces itself to

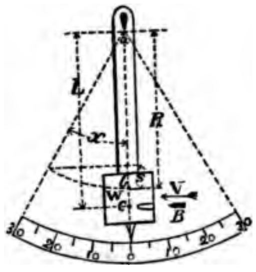
$$r+r' = \frac{l\,F\,m(F-f)}{f(F\,M+m(F-f))}. \quad . \quad . \quad . \; 16$$

It is supposed in the above arguments that the mass or ball m moves through the bore without friction or other resistance, which cannot be the case; and when that friction is taken into account the recoil will be diminished considerably. This friction and resistance to the ball in the bore acts to drag the gun with it, so that there may be no recoil until the ball leaves the muzzle, as has been confirmed by experiments with the ballistic pendulum.

§ 182. THE BALLISTIC PENDULUM.

This pendulum, represented by Fig. 215, is designed for the purpose of measuring the velocity and work of a body striking it. It consists of a long rod suspended at one end, and with a block of wood or some other soft material at the other end, forming a pendulum.

Fig. 215.



The point b is the centre of gravity of the whole pendulum, and c the centre of oscillation.

A body, or say a rifle-ball B, striking the wood block will move the pendulum an angle x and raise the centre of gravity a space s, which is the versed sine of x. The body B should strike the pendulum in a horizontal direction toward the centre of oscillation c, in order to avoid jarring in the fulcrum a.

The centre of oscillation should be determined by allowing the pendulum to swing and counting the number of single oscillations n per time t in seconds. The pendulum length L, from the fulcrum a to the centre of oscillation c, will be

$$L = \frac{39.114\,t^2}{n^2}, \text{ in inches.}$$

The centre of gravity b is found by balancing the pendulum over a sharp edge or by suspending it in different positions, as described on page 50.

The work of raising the pendulum the vertical space s is equal to the work of the striking body B.

W = weight of the pendulum in pounds.
s = space in feet which the pendulum is lifted.
B = weight of the striking body in pounds.
V = striking velocity in feet per second.

The work $\quad W s = \dfrac{B V^2}{2 g}.$

The velocity $\quad V = \sqrt{\dfrac{2 g W s}{B}}.$

The space $\quad s = R \text{ ver.sin } x.$

The angle x is measured by a graduated arc under the pendulum.

Example. The weight of the pendulum is $W = 400$ pounds, and the distance from the fulcrum to the centre of gravity $R = 10$ feet.

The weight of the striking body is $B = 0.08$ of a pound, which moves the pendulum an angle $x = 36° 53'$. Required the striking velocity of the body B?

Space $\quad s = 10 \times \text{ver.sin.} 36° \ 53' = 2$ feet.

Velocity $\quad V = \sqrt{\dfrac{2 \times 32.17 \times 400 \times 2}{0.08}} = 802.3$ feet per second.

§ 183. DYNAMICS OF HEAVY ORDNANCE.

The force of ignited gunpowder enclosed in a gun varies with the quickness of the powder, and has been found to reach as high as 40 tons to the square inch. The work of gunpowder in heavy ordnance expressed by the ordinary unit foot-pound becomes a very high number, for which a larger unit has been adopted by English ordnance officers—namely, that of foot-ton, which means a work of lifting one ton of 2240 pounds one foot high. This unit reduces considerably the number which expresses the work of heavy ordnance.

Let M denote the mass, and W the weight in pounds of a projectile of velocity V feet per second; then the work stored in the projectile will be, as before proved,

$$K = \frac{M V^2}{2} = \frac{W V^2}{2 g} \text{ in foot-pounds.} \quad . \quad . \quad . \quad 1$$

$$k = \frac{W V^2}{2\,g \times 2240} = \frac{W V^2}{144121.6} \text{ in foot-tons.} \quad . \quad . \quad 2$$

Weight of projectile $\quad W = \dfrac{144121.6\,k}{V^2}$ pounds. . 3

Velocity of feet per second $\quad V = \sqrt{\dfrac{144121.6\,k}{W}}$. . . 4

The dynamic work of different kinds of gunpowder utilized in heavy ordnance varies between 60 and 90 foot-tons per pound of powder. The average may be taken to be 80 foot-tons. Let P denote the weight in pounds of powder in a charge, then the work

$$k = 80P = \frac{W V^2}{144121.6}. \quad . \quad . \quad . \quad 5$$

Weight of charge $\quad P = \dfrac{W V^2}{11{,}500{,}000}$. 6

Weight of projectile $\quad W = \dfrac{11{,}500{,}000\,P}{V^2}$. 7

Velocity of projectile $\quad V = \sqrt{\dfrac{11{,}500{,}000\,P}{W}}$. . . . 8

Example. A gun is loaded with a charge of $P = 30$ pounds of powder for a projectile weighing $W = 200$ pounds. Required the muzzle velocity of the projectile?

Formula 49. $\quad V = \sqrt{\dfrac{11{,}500{,}000 \times 30}{200}} = 1313$ feet per second. Ans.

Quick powder in small firearms utilizes less work per weight of the explosive than does slow powder in heavy ordnance. The formulas from 47 to 49 inclusive are equally applicable for small firearms, in which the weights of the powder and projectile are expressed in grains.

$$\text{Coefficients,} \begin{cases} 11{,}500{,}000 \text{ for heavy ordnance.} \\ 8{,}000{,}000 \text{ for small firearms.} \end{cases}$$

§ 184. DYNAMIC DIAGRAMS OF HEAVY ORDNANCE.

The following diagrams are deduced from English experiments with heavy ordnance made in the year 1870. The experiments were made by an 8-inch wrought-iron gun weighing $6\frac{1}{2}$ tons, and length

of bore 126 inches. The projectile was a cast-iron cylinder 15 inches long, turned to 7.99 inches in diameter, and weighing 180 pounds.

The principal object of these experiments was to ascertain the maximum pressure of, and the effect produced by, different kinds of ignited gunpowder in the gun.

The powders experimented with were of nearly the same composition—namely, saltpetre 75, charcoal 15, and sulphur 10, making 100 parts total.

Result of the Experiments.

Nature of Powder.	Charge of powder.	Maximum pressure per square inch.	Muzzle velocity per second.
	pounds.	tons.	feet.
Rifle, large grain, R.L.G............	30	29.8	1324
Russian prismatic.....................	32	20.5	1366
Service pellet..........................	30	17.4	1338
Pebble, No. 5..........................	35	15.4	1374

The rifle large grain is a quick powder, which gave the smallest velocity with the greatest maximum pressure.

The Pebble, No. 5, is a slow powder, which gave the greatest velocity with the smallest maximum pressure.

The pressure of the ignited gunpowder was measured by Rodman's pressure-gauge. The time of operation in the gun was recorded by a chronoscope designed by Captain Andrew Noble, F. R. S., and from which the velocity of the projectile was calculated. (See *London Engineer*, Sept. 16, 1870.)

The illustration, Fig. 216, represents a gun of the Rodman pattern, and not the one used in the English experiments.

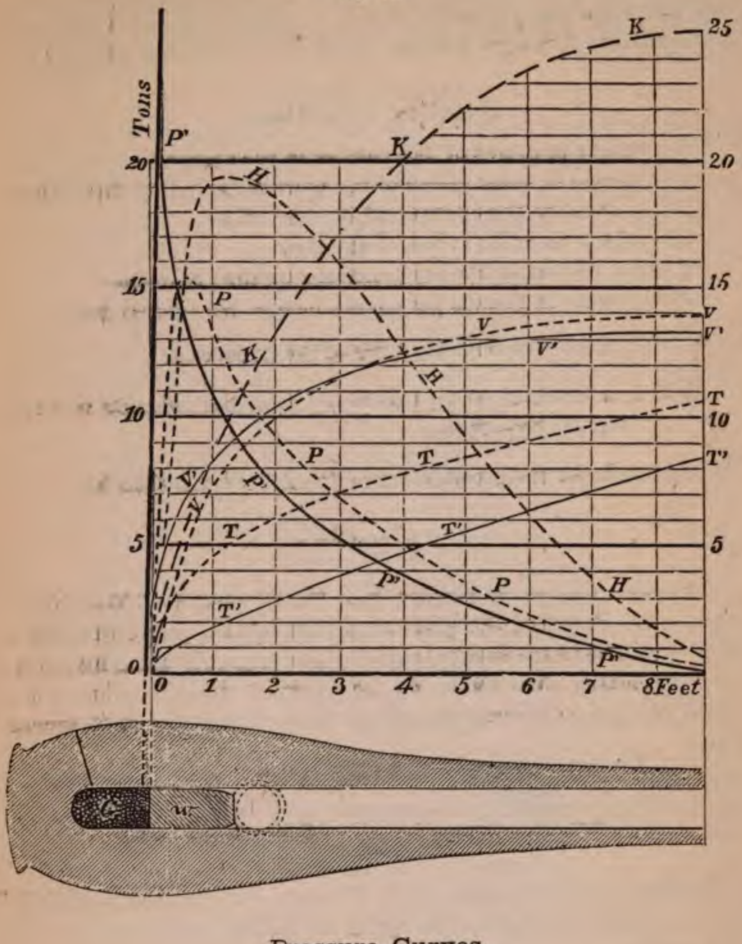
The drawn curves represent the performance with powder of rifle large grain, R. L. G., and the dotted curves that of the pebble powder, No. 5.

The absciss axis of the diagram is divided into feet, and the ordinate axis into pressures in tons per square inch.

The diagrams commence at the back end of the projectile w when ready to fire.

The figure shows the position of the projectile with 35 pounds of pebble powder. When charged with 30 pounds of R. L. G. the projectile is forced in farther until it reaches the charge.

Fig. 216.



Pressure Curves.

The drawn line P', P', P' represents the pressure curve with R. L. G., which reaches 30 tons to the square inch. The curve is not continued to that height on the diagram for want of space, but it shows that that maximum pressure is instantaneous. The ordinates measured from the base line to the curve show the pressure in tons per square inch at different positions of the projectile until it reaches the muzzle.

The dotted line P, P, P is the pressure curve for a charge of 35 pounds of pebble powder.

The pressure curves approach the shape of an equilateral hyperbola, but when the projectile attains a greater velocity in the bore the expansion of the gas does not follow up with its due pressure.

Notation of Letters.

G = weight in grains of the powder in the charge.

q = volume in cubic inches of the powder, including that, if any, between the charge and the projectile.

Q = volume in cubic inches of the bore.

q' = any volume of the bore enclosed by the projectile.

P = pressure in pounds per square inch of the ignited gas.

$\frac{G}{12}$ = square of the semi-diameter of the hyperbola.

2.55 is a constant to be subtracted from the ordinate due to a regular hyperbola.

The formula for the pressure curve P', P', P' will then be

$$P = \frac{G}{12\,q'} - 2.55. \quad . \quad . \quad . \quad . \quad 1$$

Example. Charge of powder R. L. G., 30 pounds × 700 = 210000 grains = G. Required the pressure of the ignited gunpowder when the projectile is at the first ordinate? The projectile being moved in 3 inches farther than shown on the drawing, which position is for pebble powder. The cross-area of the bore or projectile is 50 square inches.

Volume, $27 \times 50 = 1350$ cubic inches.

$$\text{Pressure,} \qquad P' = \frac{210000}{12 \times 1350} - 2.55 = 10.41 \text{ tons.}$$

The dotted pressure curve P, P, P is for a charge of 35 pounds of pebble powder, No. 5, which occupies 15 inches in the bore, or 750 cubic inches.

This being a slow powder, drives the projectile ahead and increases the volume before all of it is ignited, and thus diminishes the maximum pressure, which we find on the diagram to be when the projectile has moved about 6 inches, and when the volume is increased to $q' = 50(15+6)1050$ cubic inches. $G = 35 \times 7000 = 245000$ grains.

$$\text{Pressure,} \qquad P = \frac{245000}{12 \times 1050} - 2.55 = 16.89 \text{ tons.}$$

The diagram shows only 15.4 tons, being slow powder.

The treatment of pressure, volume, temperature and work of ignited gunpowder belongs to dynamics of heat, from which the Formula 1 is borrowed.

Work Curve.

The work accomplished by the charge is represented by the area bounded within the pressure curve and abscissa to the ordinate of the pressure. This area, multiplied by the 50 square inches section of the bore, gives the work in foot-pounds accomplished by the ignited powder.

The work has been calculated for each ordinate, and set off from the abscissa to the work curve K, K, K.

The number on the scale, multiplied by 100, gives the work accomplished in foot-tons. It is this work which is stored in the projectile when set in motion—namely,

$$K = \frac{w V^2}{2 g} \text{ in foot-pounds. } \quad . \quad . \quad . \quad . \quad 2$$

$$K = \frac{w V^2}{4480 g} \text{ in foot-tons. } \quad . \quad . \quad . \quad . \quad 3$$

$w = 180$ pounds, weight of the projectile.

$V = 1374$ feet per second, the muzzle velocity of the projectile with pebble powder, No. 5.

$$\text{Work} \quad K = \frac{180 \times 1374^2}{4480 \times 32.17} = 2357.85 \text{ foot-tons.}$$

The diagram shows 2500 foot-tons accomplished by the charge, and $2500 - 2358 = 142$ foot-tons, which must have been expended in friction and leakage in the bore.

The diagram can, however, not be very correct.

Theoretical Work in Ordnance.

S = length in feet of the bore of the gun.

s = length occupied by the powder.

s' = length passed through by the projectile in the bore.

A = cross-area of the bore in square inches.

The differential work will then be

$$\partial K = A\, P \partial s' = A\left(\frac{G}{12\, q} - 2.55\right)\partial s'. \quad . \quad . \quad . \quad 4$$

$$K = A\int \frac{G\, \partial s'}{12\, q} - A\int 2.5\ \partial s'.$$

As the bore is cylindrical, $\delta s' = \delta q$, and by integrating the formula from s to S, the work will be

$$K = A\left[\frac{G}{12}\,hyp.log.\frac{Q}{q} - 2.55(S-s)\right] \quad . \quad . \quad . \quad . \quad 5$$

This work should be equal to that stored in the projectile when leaving the muzzle, or

$$K = \frac{wV^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 6$$

Having given the charge G and the dimensions of the gun, the velocity of the projectile should be

$$V = \sqrt{\frac{2gA}{w}\left[\frac{G}{12}\,hyp.log.\frac{Q}{q} - 2.55(S-s)\right]} \quad . \quad . \quad . \quad 7$$

The velocity will be slightly less on account of friction and leakage in the bore.

In a gun of 8 inches in diameter of bore the cross-area is $A = 50$ square inches, and $q' = 12 \times 50 = 600$ cubic inches per foot in the bore; then

$$s' = \frac{q'}{600}, \text{ and } q' = 600 \cdot s'. \quad . \quad . \quad . \quad . \quad . \quad 8$$

$$K = \frac{G}{144}\,hyp.log.\frac{S}{s} - 127.5(S-s). \quad . \quad . \quad . \quad . \quad 9$$

This is the formula for work of gunpowder in an eight-inch gun.

For the pebble powder we have $G = 245000$ grains. The length of the gun $S = 10.5$ feet, and the length of charge $s = 1.25$ feet. Required the work of the charge?

$$K = \frac{245000}{144}\,hyp.log.\frac{10.5}{1.25} - 127.5(10.5 - 1.25) = 2535 \text{ foot-tons.}$$

The theoretical work is thus 35 foot-tons more than the graphical work shown by the diagram, the reason of which is that the pressure curve P, P, P does not fill up the space to the ordinate axis, because the projectile moved before the due maximum pressure was reached. The calculation, however, proves the correctness of the experiments.

Time Curves.

The time curves T', T', T' for R. L. G. powder, and T, T, T for the pebble, No. 5, were obtained by Noble's chronoscope, which recorded the moment the projectile passed each ordinate. The numbers on the scale divided by 1000 give the time in decimals of a second in which the projectile reached each ordinate.

The muzzle time for the pebble powder is shown by the diagram to be about $\frac{10.6}{1000} = 0.0106$ of a second. The fifth ordinate time for the R. L. G. powder is $\frac{5.5}{1000} = 0.0055$ of a second.

Velocity Curves.

The velocity curves V', V', V' for the R. L. G. and V, V, V for the pebble powder were obtained by comparing the time and space.

$$\text{Velocity} \quad V = \frac{\partial L}{\partial T}. \quad . \quad . \quad . \quad . \quad 10$$

The velocity of the projectile in any part of the bore can be approximated by Formula 7 by placing S and s' for Q and q.

$$V = \sqrt{\frac{64.34\ A}{w}\left[\frac{G}{12}\ hyp.log.\ \frac{s'}{s} - 2.55(s' - s)\right]}. \quad . \quad 11$$

It is supposed in this formula that the diameter of the chamber or bore occupied by the powder is equal to that occupied by the projectile.

s = length in feet of the chamber or bore occupied by the powder.
s' = distance from the bottom of the bore to the projectile where the velocity is required.

The starting velocity is slightly affected by different compositions of powder, but the muzzle velocity will agree with the formula.

The number on the scale of the diagram multiplied by 100 gives the velocity in feet per second of the projectile.

It will be observed that the velocity curves tangent the abscissas at the muzzle, which proves that the velocity is no more increased by the charge, and that the pressure of the charge is reduced to almost nothing at the muzzle proves that the gun is of the proper length.

We may estimate the proper length of a gun as follows:

$$P=\frac{G}{12\,Q}-2.55=0,\quad \text{of which } \frac{G}{12\,Q}=2.55,\quad \text{and } Q=\frac{G}{12\times 2.55}=\frac{G}{30.6}.$$

Q = volume of the bore in cubic inches.
D = diameter of the bore in inches.
S = length of the bore in feet.
$Q = 0.942\, D^2 S.$

Length of gun, $S=\dfrac{G}{28.8\,D^2}.$

Horse-power Curve.

The horse-power curve H, H, H is obtained by multiplying the force or pressure by the velocity of each ordinate, and the product multiplied by $\dfrac{2240 \times 50}{550} = 203.636.$

The number on the scale multiplied by 100,000 gives the acting horse-power at that ordinate. The maximum power for the pebble powder is over 1,900,000 horses.

The work and horse-power curves were not given on the English diagrams.

The six fundamental principles of dynamics are thus illustrated in the performance of ordnance—namely,

Elements.		*Functions.*	
Force	$F.$	Space	$S = V\,T.$
Velocity	$V.$	Power	$F = F\,V.$
Time	$T.$	Work	$K = F\,V\,T.$

The English experiments with different kinds of gunpowder show the importance of applying the science of dynamics to the performance of heavy ordnance.

The quick powder, R. L. G., gave an instantaneous maximum pressure of nearly 30 tons to the square inch, whilst the slow pebble, No. 5, produced a continuous maximum pressure of only 15.4 tons, and gave a greater velocity to the projectile per weight of powder.

The instantaneous maximum pressure is of little or no use in propelling the projectile, but it injures the metal in the chamber and tends to burst the gun.

It appears that the instantaneous and excessive maximum pressure overstrains the gas so that it loses much of its elasticity, as indicated by the diagram. The pressure above 20 tons to the square inch rises and falls nearly in the same vertical line.

The best gunpowder for heavy ordnance is that which gives the greatest area of work with the least maximum pressure.

The charge ought to be arranged with powder of different quickness, so that a slow powder nearest to the projectile is first ignited, than a quicker powder, until the quickest at last; which would generate a low and continued maximum pressure with greater area of work, and which would give much greater velocity of the projectile with less risk of bursting the gun.

With such arrangement of powder in the charge the gun could be made longer and lighter at the bridge.

The best charge of powder is that which gives the greatest radius of curvature of the pressure curve at the maximum pressure.

§ 185. GUNPOWDER PILE-DRIVER.

This pile-driver, invented by Thomas Shaw of Philadelphia, is worked by gunpowder as motive-power; it consists of a gun a placed on a pile b, and a ram c with a plunger d fitting closely in the bore of the gun. The ram is guided by a high framing, as represented by Fig. 217.

Fig. 217.

The explosion of the gunpowder in the gun, drives the plunger d with the ram c to a considerable height, and the recoil of the gun drives the pile into the ground.

Whilst the ram is up a new charge is placed in the gun, and in the fall of the ram the plunger enters the bore and compresses the enclosed air to a high pressure and heat, which ignites the charge for another explosion for driving up the ram; and so the operation is continued until the pile is driven home to its destination. The pile is also driven into the ground by the compression of the air in the gun, but the greatest drift is in the recoil.

The ram can be held at any height by a friction-brake extending the whole height of the framing.

At the top of the framing is a cushioning piston e, which fits closely in a bore in the ram, for the purpose of preventing the ram from striking or rising above the limited height.

The bores in the gun and ram are funnel-shaped at the top for admitting the plunger and piston with safety from striking the edges.

A very slow gunpowder should be used in the charge.

§ 186. THEORY OF THE GUNPOWDER PILE-DRIVER.

The recoil of the gun or set of the pile will be the same as that by the Formula 16, page 253.

$$\text{The set } S = r + r' = \frac{L\,F\,w(F-f)}{f(FW + w(F-f))}. \qquad 1$$

S = set of the pile in feet.

L = length in feet of the bore in the gun passed through by the plunger, omitting the funnel.

F = mean force in pounds of the gunpowder explosion.

w = weight in pounds of the ram and plunger.

W = weight in pounds of the gun and pile.

f = force of resistance in pounds to the pile in the ground, less the weight W.

Call R = actual resistance to the pile, then $R = W + f$, and $f = R - W$.

The efficiency of the pile-driver consists in obtaining great recoil, and the formula shows that the greater weight of ram and the greater length of bore the greater will be the set S on recoil; but the bore should not be made longer than is necessary for the gases to raise the ram a sufficient height for igniting the charge in its fall.

The force of gunpowder cannot be correctly ascertained without experiments, and different compositions of powder give different diagrams of force and expansion. This subject belongs to dynamics of heat, from which the following three formulas are taken. They give an approximate value of the performance of gunpowder.

G = weight in grains of powder in the charge.

Q = volume in cubic inches of the gas of the exploded gunpowder.

P = pressure in pounds per square inch of the gas.

L = length of the bore of the gun in feet.

l = length of bore occupied by the charge.

F = mean force of the charge in pounds.

$$\text{Pressure} \quad P = \frac{100\,G}{Q}. \qquad 2$$

$$\text{Work} \quad K = 100\,G\ hyp.log.\left(\frac{L}{l}\right) \qquad 3$$

$$\text{Mean force} \quad F = \frac{100\,G}{L-l}\ hyp.log.\left(\frac{L}{l}\right) \qquad 4$$

Example. Assume the cross-area of the bore in the gun to be 40 square inches, and acted upon by the explosion of $G = 700$ grains of powder. Length of the bore $L = 2$ feet and $l = 0.08333$ of a foot, which is one inch, the distance between the plunger and the bottom of the gun at the time of explosion. Required the pressure P and P' per square inch in the gun at the time of explosion and when the piston leaves the muzzle?

The volumes of gas are $Q = 1 \times 40 = 40$ cubic inches at the time of explosion, and $Q' = 40 \times 24 = 960$ cubic inches, the volume of the bore.

$$\text{Pressure} \quad P = \frac{100 \times 700}{40} = 1750 \text{ pounds per square inch.}$$

$$\text{Pressure} \quad P' = \frac{100 \times 700}{960} = 73 \text{ pounds per square inch.}$$

Required the work done by the explosion?

$$\text{Work} \quad K = 100 \times 700 \times hyp.log\left(\frac{2}{0.0833}\right) = 222460 \text{ foot-pounds.}$$

Required the mean pressure F?

$$\text{Mean pressure} \quad F = \frac{K}{L - l} = \frac{222460}{2 - 0.0833} = 116066 \text{ pounds.}$$

The work K of the explosion of the gunpowder performs two duties—namely, to drive the pile into the ground and raise the ram.

h = the height in feet to which the ram is driven, the work of which is $w\,h$.

The work of driving the pile into the ground is $S f$. Then we have the work

$$K = w\,h + Sf, \qquad \text{of which } f = \frac{K - w\,h}{S}, \quad . \quad . \quad . \quad 5$$

and the resistance to the pile in the ground will be

$$R = f + W = \frac{K - w\,h}{S} - W. \quad . \quad . \quad . \quad 6$$

The bearing capability of the pile should be calculated from Formula 5, after the last blow or set S.

The quality of the gunpowder can be ascertained experimentally by placing the gun on a solid foundation, so that there will be no recoil of the explosion.

$$\text{Then,} \qquad F(L - l) = w\,h, \quad . \quad . \quad . \quad . \quad 7$$

$$\text{and the mean force,} \qquad F = \frac{w\,h}{L - l}. \quad . \quad . \quad . \quad . \quad 8$$

§ 187. COMPRESSION OF AIR IN THE GUN.

When the ram falls and drives the plunger into the bore of the gun, the air will be compressed to a small volume and heated to a high temperature. Knowing the weight of the ram and height of its fall, it is required to find the compression and temperature of the air in the gun. For this purpose we must borrow some formulas from dynamics of heat, which do not belong to this treatise:

p = pressure of the air on the plunge-piston when entering the bore, which is 14.7 pounds to the square inch.
P = pressure of the compressed air on the plunger.
V = volume of air of atmospheric pressure in the bore.
v = volume of the compressed air.

$$\text{Then,} \qquad P = p\left(\frac{V}{v}\right)^{1.4} \quad . \quad . \quad . \quad . \quad . \quad 9$$

The bore of the gun is cylindrical, and the enclosed volume of air is therefore as the distance between the plunger and the bottom of the gun.

$$\text{Then,} \qquad P = p\left(\frac{L}{l}\right)^{1.4} \quad . \quad . \quad . \quad . \quad . \quad 10$$

The work accomplished by compressing the air will be

$$\partial K = P\partial l = p\left(\frac{L}{l}\right)^{1.4}\partial l. \quad . \quad . \quad . \quad 11$$

$$K = p\int\frac{L^{1.4}}{l^{1.4}}\partial l = p\frac{L^{1.4}}{0.4\, l^{0.4}}. \quad . \quad . \quad . \quad . \quad 12$$

This work is wholly consumed by the generation of heat in the compressed air. In case the work of the falling of the ram $w\,h$ is greater than $p = \frac{L^{1.4}}{0.4\, l^{0.4}}$, the difference will be utilized in driving the pile into the ground. This set of the pile is not included in S, Formulas 5 and 6.

P and p mean the actual pressure above vacuum of the enclosed air on the whole piston, from which must be subtracted the atmospheric pressure on the ram-plunger, which is 14.7 A, when A = area in square inches of the plunger piston.

The work done by the ram in the compression of the air will then be

$$w\,h = p\frac{L^{1.4}}{0.4\,l^{0.4}} - 14.7\,A(L-l), \quad . \quad . \quad . \quad 13$$

or $$w\,h = 14.7\,A\left(\frac{L^{1.4}}{0.4\,l^{0.4}} + l - L\right); \quad . \quad . \quad . \quad 14$$

and if we include the work that may set the pile during the compression of the air, which is $f\,s$, we have

$$w\,h = 14.7\,A\left(\frac{L^{1.4}}{0.4\,l^{0.4}} + l - L\right) + f\,s. \quad . \quad . \quad . \quad 15$$

The set $$s = \frac{1}{f}(w\,h - 14.7\,A)\left(\frac{L^{1.4}}{0.4\,l^{0.4}} + l - L\right). \quad . \quad . \quad 16$$

§ 188. **TEMPERATURE OF THE COMPRESSED AIR IN THE GUN.**

T = temperature of the compressed air.

t = temperature of the air or gas of atmospheric pressure in the gun when the plunger enters the muzzle.

The absolute zero = 461° below zero of Fahrenheit's scale.

$$\frac{L}{l} = \left(\frac{461+T}{461+t}\right)^{2.45}. \quad . \quad . \quad . \quad . \quad . \quad 17$$

$$T = \sqrt[2.45]{\frac{L}{l}}(461+t) - 461. \quad . \quad . \quad . \quad . \quad 18$$

It is supposed in these formulas that no air leaks out during the compression, and that no heat is conducted from the air by the metals enclosing it; which cannot be the case, and for which reason a deduction of at least 25 per cent. should be made in the length L.

The temperature of ignition of slow gunpowder may be assumed to be $T = 600°$ Fahr., and assuming the temperature of the air or gas in the gun under atmospheric pressure to be $t = 80°$, we have $461 + 600 = 1061°$, and $461 + 80 = 541°$, the absolute temperatures.

Then $$\left(\frac{1061}{541}\right)^{2.45} = 5.208 = \frac{L}{l}.$$

$$L = 5.208\,l, \text{ and } l = 0.192\,L,$$

when ignition of the gunpowder takes place, but if 25 per cent. is deducted for leakage and conduction of heat, we have

$$L = 6.944\,l, \text{ and } l = 0.144\,L.$$

The following data of performance of the gunpowder pile-driver has been tabulated by F. C. Prindle, C. E., U. S. N., from records of work done at League Island on the river Delaware:

No. of Record.	No. of Piles Driven.	Diameter of Piles. Inches. Top. Max.	Top. Min.	Top. Av.	Bottom. Max.	Bottom. Min.	Bottom. Av.	Distance Driven. Feet. Max.	Min.	Av.	No. of Blows per Pile. Max.	Min.	Av.	Weight of Powder per Pile. Pounds. Max.	Min.	Av.	Weight of Ram. Pounds.	Bore of gun. Inches.
1	811	1	10.5	13.1	13	6	8.7	22.5	14	19.4	19	3	5.2	1.3	$\frac{1}{4}$	$\frac{1}{8}$	1300	$6\frac{1}{4}$
2	966	15	10	12	11	6.5	8	30	21	24	...	...	20	...	...	2	1200	$5\frac{3}{4}$
3	457	19	9	12.2	14	7.5	8.7	36	20	29.2	85	11	30.4	$9\frac{1}{4}$	$1\frac{1}{4}$	$3\frac{3}{4}$	1700	$6\frac{3}{4}$
4	63	16	10	12.7	12	7.5	9	31	25	29.5	122	39	59.6	15	4	$6\frac{1}{8}$	1200	$5\frac{3}{4}$
5	172	17.5	9	11.4	12	7	8.3	32.5	26	29.2	30	6	12.7	$4\frac{7}{8}$	1	3.2	2170	$7\frac{1}{8}$

Records Corresponding to the Numbers in the Table.

Record 1. The first machine employed upon actual work. The framing was made of cast-iron, mounted on a scow and operated afloat at the landing wharf, Fig. 218. The piles of heavy yellow pine driven through mud containing clay to compact gravel.

Record 2. The framing of wood and iron, mounted on land. Piles of hemlock, and firmly driven without pointing, through stiff clayey material, mixed with sand, to hard gravel and boulders. The piling was for the foundation for storehouses, etc. Number of blows and weight of powder approximate.

Record 3. The same machine operating, in the same kind of ground as in Record 2. Foundation for iron-plating shop.

Record 4. The same machine as in Records 2 and 3, but with lighter ram.

Record 5. The new improved machine, with wrought-iron framing, completed the work.

Fig. 218.

SHAW'S PATENT GUN POWDER PILE-DRIVER.

This illustration, Fig. 218, represents Mr. Shaw's gunpowder pile-driver on a scow in the Delaware River at League Island Navy-Yard.

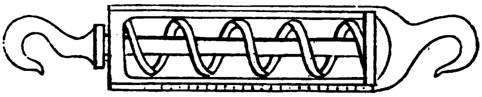
§ 189. DYNAMOMETERS.

Dynamometers are instruments for measuring force, power and work. The simplest form of dynamometer is that of a spring.

§ 190. SPRING DYNAMOMETER.

The construction of this dynamometer is readily understood by Fig. 219. It is graduated by experiments in compressing the spring with known weights.

Fig. 219.

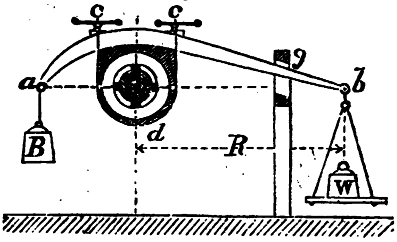


This dynamometer is best adapted for measuring the force of pulling a load on a road, a boat on a canal, or of towing a ship. The force in pounds indicated by the dynamometer, multiplied by the velocity in feet per second, will be the power in effects, which divided by 550 will give the horse-power in operation.

§ 191. PRONY'S FRICTION DYNAMOMETER.

This dynamometer consists of a friction brake, as shown by the illustration. It is keyed on the shaft A, which transmits the power and work to be measured.

Fig. 220.



The lever of the brake should be balanced at B before the weight W is put on the scale, and if it is not balanced, the weight of the lever and scale should be weighed at the scale and added to the weight W.

The weight W on the scale is the force acting on the lever or radius R.

It is supposed that all the power and work transmitted by the shaft is consumed by the friction in the brake. When the shaft is running with its average speed of n revolutions per minute, the strap is tightened up with the screws, so that the lever will barely lift the weight W, which is also adjusted to suit the motion. When the weight and friction are well balanced, count the revolutions per minute of the shaft.

The power transmitted through the shaft is equal to the weight W multiplied by the velocity of the circumference of the radius R, making the same revolutions as the shaft.

The velocity in feet per second is

$$V = \frac{2\pi R n}{60}.$$

Power $$P = W V = \frac{2\pi R n W}{60} \text{ in effects,}$$

which divided by 550 give the

Horse-power $$\text{HP} = \frac{2\pi R n W}{60 \times 550} = \frac{W R n}{5252.2}.$$

The work K in foot-pounds consumed by the friction of the brake in the time T in seconds will be

Work $$K = \frac{2\pi R n W T}{60} = \frac{W R n T}{9.55}.$$

All this work consumed by the friction is restored by generating heat, which makes the brake so hot that a constant stream of water must run on it to absorb the heat whilst the experiment is made, otherwise the wood in the brake would take fire.

When convenient it is best to make the lever $R = 10.5$ feet, or 10 feet 6 inches, which will make the circumference 66 feet; in which case, the horse-power will be

$$\text{HP} = \frac{66\, n W}{550 \times 60} = \frac{2\, n W}{1000}.$$

That is to say, the product of the revolutions per minute and weight W, multiplied by 2 and point off three places, will be the horse-power of the experiment.

A lever of $R = 5$ feet 3 inches will make the circumference 33 feet, and the horse-power

$$\text{HP} = \frac{n W}{1000}.$$

§ 192. BEVEL-WHEEL DYNAMOMETER.

Fig. 221 represents a side elevation and Fig. 222 a plan of the dynamometer.

Fig. 221.

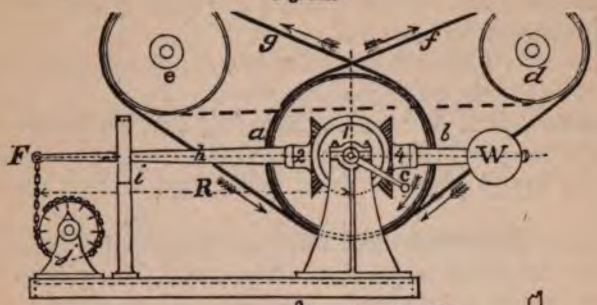
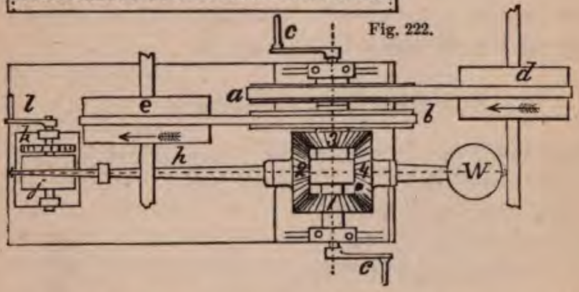


Fig. 222.



It can be worked either by cranks c, c, or by pulleys a and b.

Let d be a shaft and pulley communicating motion by the dotted belt to the shaft and pulley e, and it is required to measure the power transmitted between the two shafts. Place the dynamometer so that it takes a belt f from the pulley d on its pulley a, and another belt g from the pulley b to the pulley e. The pulley a and the bevel-wheel l are fastened on the crank-shaft.

The bevel-wheel 3 is fast on the pulley b, but both are loose on the crank-shaft.

The bevel-wheels 2 and 4 are both loose on the arm h. Either one of the wheels 2 and 4 could be dispensed with, but the dynamometer works better with the two wheels. Now set the pulley and shaft d in motion in the direction of the arrow, and all the power will be transmitted through the dynamometer to the pulley e. The problem is to measure the power transmitted.

The arm h is fitted loose on the crank-shaft as a fulcrum, around which the arm is allowed a small angular motion, limited by the slot in the pillar i.

When power is transmitted the bevel-gear acts to lift the arm h, but a force F is applied to keep the arm in a horizontal position. If the pulley b with the wheel 3 is held stationary, and the arm be allowed to revolve, it would make one revolution whilst the crank-shaft makes two; but when the pulley b and wheel 3 revolve the arm h is held horizontal by a force F.

The force F is derived from a spiral spring enclosed in the drum j, which acts on the chain like that in a watch.

A ratchet-wheel is fastened on the spring axis, and by the aid of the crank l the spring can be regulated to the exact force required to balance the arm h with the power transmitted through the dynamometer.

The drum j and ratchet-wheel k are both graduated to indicate the force F on the chain in pounds. The sum of the readings is the force on the chain.

The weight W is for balancing the arm h.

Notation of Letters.

D = diameter in feet of each of the pulleys a and b.
f = force of tension in pounds on the belts.
R = radius of the arm h in feet.
F = force in pounds on the chain.
The static momentums of the combination will then be

$$F : f = D : R, \quad \text{and} \quad F R = f D.$$

$$F = \frac{f D}{R}, \quad \text{and} \quad f = \frac{F R}{D}.$$

When the dynamometer is worked by the cranks without the belts and pulleys,
r = radius in feet of the crank c.
f' = force in pounds on the crank-pin.

$$F : f' = 2 r : R, \quad \text{and} \quad F R = f' \, 2 r.$$

$$F = \frac{2 f' r}{R}, \quad \text{and} \quad f' = \frac{F R}{2 r}.$$

Transmission of Power.

V = velocity in feet per second of the belts.
v = velocity of the end of the arm h if allowed to swing.
n = number of revolutions per minute of the crank-shaft.

$$V : v = D : R, \text{ and } V R = v D.$$

$$V = \frac{v\, D}{R}, \quad \text{and} \quad v = \frac{V R}{D}.$$

$$V = \frac{\pi\, D\, n}{60}, \quad \text{and} \quad v = \frac{\pi\, R\, n}{60}.$$

Power $\quad P = f V = F v$ in effects.

Power $\quad P = \dfrac{f \pi\, D\, n}{60} = \dfrac{F \pi\, R\, n}{60}.$

The power transmitted through the dynamometer will then be

$$P = \frac{F \pi\, R\, n}{60} = 0.05236\, F\, R\, n.$$

$$\text{Horse-power } \mathrm{HP} = \frac{F \pi\, R\, n}{60 \times 550} = \frac{F\, R\, n}{10503.55}.$$

When the dynamometer is working we have given the force F from the spring graduation. The length R of the arm h is given and constant for each dynamometer. The number of revolutions per minute is obtained by counting the same. We have thus given all that is necessary for calculating the power transmitted through the dynamometer.

In constructing a dynamometer of this kind it would be best to give such length to the lever R as to make an even number in the denominator of the formula; for instance, if $R = 5.251775$ feet, we have the

$$\text{Horse-power} \quad \mathrm{HP} = \frac{F\, n}{2000}.$$

Allowing five per cent. for friction in the dynamometer, a five-feet lever would make the formula the same as above.

Machinery very rarely transmits power uniformly from one locality to another; which is particularly the case with the ordinary steam-engine, as has already been explained.

The storage and delivery of work by a fly-wheel causes an irregularity in the power transmitted, which can be measured by the dynamometer.

The oscillation of the arm h and spring-drum j indicates this irregularity, which can be read on the graduation, and thus enables us to determine with great precision the efficiency of a fly-wheel. Suppose the drum and ratchet-wheel to indicate a mean force $F = 150$ pounds

when the drum oscillates a difference 12 pounds; the irregularity will then be

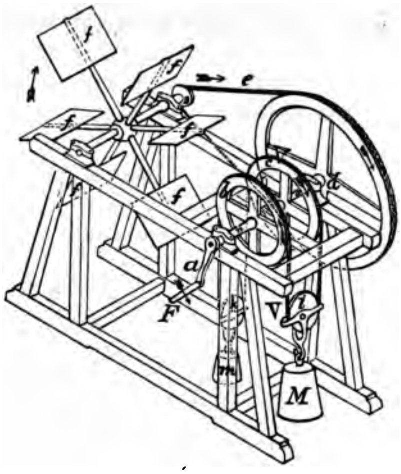
$$=\frac{\frac{1}{2}\times 12}{150}=0.04, \text{ or 4 per cent.}$$

A dynamometer of this kind would be very useful in institutions where the subject of dynamics is taught. The students should be made to work the cranks of the dynamometer, and calculate their own force, power and work, and thus learn practically how to distinguish the different elements and functions in dynamics, how they bear upon one another, and to conceive real magnitudes of such quantities.

The dynamometer could easily be arranged with indicators by which to trace diagrams of the different elements and functions involved in the operation.

§193. DYNAMOMETER AT THE ROYAL TECHNOLOGICAL INSTITUTE, STOCKHOLM.

Fig. 223.



The accompanying illustration represents an isometric perspective view of the dynamometer at the Royal Technological Institute, Stockholm. The power is applied on the crank *a*, and communicated through the pulley *b*, rope *V*, pulley *c*, wheel *d*, rope *e*, rope-pulley *g*, and is consumed by the fans *f f f*. The shafts *n* and *o* are in one line, but not connected between the pulleys *b* and *c*. The endless rope *V V* is held tight in the grooves of the pulleys *b* and *c* by means of

two weights w and W, as will be understood by the drawing. If the two weights were alike, they could communicate no motion to the pulleys, but suppose $w=10$ pounds, and $W=20$ pounds, then there would be 10 pounds more weight on the sheave i, than on the sheave k, of which five pounds would pull on each pulley b and c. Let the radius of the crank a be equal to the radius of the pulleys, then it would require a force of five pounds to turn the crank in the direction of the arrow. If the crank is turned with an irregular velocity, it would only raise or lower the weights, but a constant force of five pounds would always act on the pulley c to communicate motion to the fans. The power in operation will be equal to the force multiplied by the velocity of the rope V, and the work accomplished will be equal to the power multiplied by the time of operation.

Notation of Letters.

R = radius of the crank in feet, which we have supposed to be equal to the radii of the pulleys b and c.

F = force in pounds acting on the crank a.

V = velocity in feet per second of the rope V, which, in the supposed case, will be equal to that of the crank-pin.

T = time of operation in seconds.

W and w = weights on the pulleys b and c in pounds.

P = power in dynamic effects, of which there are 550 per horse-power.

K = work, in foot-pounds of work.

n = number of revolutions per minute of the pulley c.

Then we have

The force $F=\frac{1}{2}(W-w)$, and velocity $V=\dfrac{2\pi R n}{60}$.

Power $P=F\,V$, and work $K=F\,V\,T$.

Example. Radius of the crank or pulleys, $R=1.25$ feet, making $n=28$ turns per minute. $W=80$, and $w=20$ pounds. Required, the force $F=?$, velocity $V=?$, power $P=?$, and how much work, $K=?$, will be accomplished in one hour, or $T=3600$ seconds?

Force $F=\frac{1}{2}(80-20)=30$ pounds.

Velocity $V=\dfrac{2\times 3.14\times 1.25\times 28}{60}=3.66$ feet per second.

Power $P=30\times 3.66=109.9$ effects.

Work $K=109.9\times 3600=3956.4$ foot-pounds.

The average power of a man working eight hours per day is 55 effects, which will be an accomplished work of $K = 55 \times 8 \times 3600 = 1584000$ foot-pounds in a day's work.

In order to regulate the velocity to suit the power, the dynamometer has an arrangement by which to set the fans at any desired angle while in motion, which arrangement is not shown on the drawing. Students used to work the dynamometer in a spirit of emulation to outdo each other in power and work. Some could accomplish the greatest power, and work with less force and more velocity, whilst others preferred more force and less velocity.

Arrangements could easily be made to register on the dynamometer the force, velocity, power, and work in the form of diagrams.

Dynamometers of this kind ought to be employed in all scientific institutions where dynamics are taught, for we have yet no better means by which to imbue the student with the real substance of dynamics. Any student who has worked this instrument for a few hours will probably not commit the error of saying that *work is independent of time, or that time* is included in power, which erroneous ideas are yet maintained in text-books.

This dynamometer does not only teach the student the different properties of *force, power* and *work*, but it enables him to conceive and compare, with great precision, real magnitudes of those quantities, which is of great importance in designing machinery.

In the year 1850 the author made a great many experiments with different kinds of screw-propellers, in which was employed a dynamometer of this description, made by Thomas Mason of Philadelphia, and which gave great satisfaction; and he has always considered it the best form of dynamometer where it can be conveniently applied.

§ 194. DYNAMICS OF SOUND.

Sound is work, consisting of the three simple elements $F\ V\ T$, of which F = force of the sound, V = velocity of vibration, and T = time of continuance of the sound.

The loudness of sound is

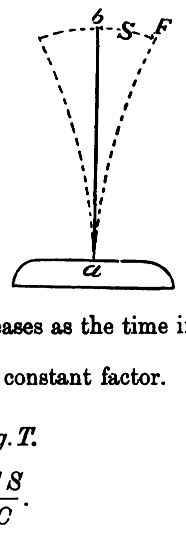
$$\text{Power} \qquad P = F\ V.$$

The pitch of the sound indicates the proportion between F and V.

Two different sounds of different pitch may be of equal power or loudness, but the high-pitch sound is then produced by small force F and great velocity V, whilst the low-pitch sound is produced by greater force F' and smaller velocity V', so that the products $F\ V$ and $F'\ V'$ are alike in the two sounds.

Let an elastic spring $a\ b$ be drawn aside a space S where the force is F. Leave the spring to take its own course, and it will move fore and back and set the surrounding air into vibration, which produces sound.

Fig. 224.



The work expended in drawing the spring aside is $K = F\ S$, which work is restored by producing sound, and the spring will continue to sound until all the work $F\ S$ is consumed.

The loudness or power $P = F\ V$ of the sound is greatest at the start of vibration, after which it will gradually diminish, and finally fade away to nothing.

Differential work $\partial k = P \partial t$, but the power decreases as the time increases that we can place $P = \frac{C}{t}$, in which C is a constant factor.

$$\partial k = \frac{C \partial t}{t}, \qquad K = \int \frac{C \partial t}{t} = C\ hyp.log.\ T.$$

$$\text{Time of sound, } hyp.log.\ T = \frac{K}{C} = \frac{F\ S}{C}.$$

The force F of an elastic spring is as the space S within the limit of vibration, and the mean force in the space S is therefore $\frac{1}{2}\ F$. The spring vibrates the same pitch of sound in any space S, and the proportion between F and V is therefore constant. The loudness of the sound or power $P = F\ V = C\ S^2$, or the power of the sound, is as the square of the space of vibration. The velocity V in the above formulas means that of the force producing the sound, and not the velocity of the sound from the sonorous body.

Noise in Machinery.

All sound or noise in operating machinery represents so much work lost, and the loudness of the noise represents the power lost. The noise in a cotton-mill, rolling-mill, or railroads, etc. represents power and work lost.

Noise of a Steam-hammer.

A steam-hammer falling on its bare anvil will set the whole system, with the surrounding air, into vibration, and create a great noise, which represents the whole work of the hammer; but the same hammer falling upon a puddle-ball, or upon white-hot iron, will create very little noise, and the greatest part of the work of the hammer is then utilized in forging the iron.

Report of a Gun.

The report of a gun represents the work lost of the total work due from the ignited powder. If the ignited gas in the gun could be allowed to propel the projectile until its force of expansion is reduced to the atmospheric pressure, there would be no loud report.

Sound of a Bell.

The work of a clapper in striking its bell is represented by the sound produced. M = mass of the clapper and V = velocity of the strike. Then the

$$\text{Work of the clapper} = \frac{MV^2}{2} = \text{work of the sound.}$$

Sound of Men and Animals.

The sound of men or animals is produced by the heat in the body, which is work. A speaker, singer, or a musician playing a wind instrument draws from his store of heat as when doing hand labor, and the louder he speaks, sings, or plays, the greater power is drawn from his heat, and he ultimately becomes fatigued as when working a crank.

Music of an Organ.

The weight on the organ-bellows multiplied by the velocity with which it sinks whilst air is discharged and none enters is the power in effects, which, divided by 550, gives the horse-power of the organ. The same weight multiplied by the space it sinks is the work done by the organ in producing the music.

Dynamics of Sound includes the science of acoustics, and is a very extensive and interesting subject, which requires a separate treatise.

§ 195. VELOCITY OF SOUND IN AIR.

The velocity of sound in air has been determined both by experiments and theory by Newton, Laplace, Dalton and various others; the summary of which is

$$V = 1089.42\sqrt{1 + 0.00208(t - 32)}.$$

$$D = 1089.42\,T\sqrt{1 + 0.00208(t - 32)}.$$

V = velocity in feet per second of the sound.
t = temperature Fahr. of the air.
D = distance in feet travelled in the time T in seconds.
1089.42 = velocity of sound at 32°, temperature of the air.
0.00208 = volume of expansion per degree Fahr. of the air.

Distance in Feet which Sound Travels in Air at Different Temperatures

Time	Temperature of the Air, Fahrenheit Scale.										
sec.	0°	10°	20°	32°	40°	50°	60°	70°	80°	90°	100°
1	1000	1064.2	1075.7	1089.4	1098.5	1109	1120	1131	1142	1153	1164
2	1985	2128	2151	2179	2197	2219	2241	2262	2285	2306	2328
3	2978	3193	3227	3268	3295	3328	3361	3393	3427	3459	3492
4	3971	4257	4303	4358	4394	4438	4482	4524	4570	4613	4656
5	4964	5321	5378	5447	5492	5548	5603	5655	5712	5766	5821
6	5956	6385	6454	6536	6591	6657	6723	6786	6855	6919	6984
7	6949	7449	7530	7626	7689	7767	7844	7917	7997	8072	8148
8	7962	8514	8606	8715	8788	8876	8964	9049	9140	9225	9312
9	8934	9578	9681	9805	9886	9986	10085	10180	10282	10379	10476
10	9927	10642	10757	10894	10985	11096	11306	11311	11425	11532	11640
11	10920	11706	11833	11983	12083	12205	12326	12442	12567	12685	12804
12	11912	12770	12908	13073	13182	13315	13447	13573	13710	13838	13968
13	12905	13835	13984	14162	14280	14424	14567	14704	14852	14991	15132
14	13898	14899	15060	15252	15379	15534	15688	15835	15995	16145	16296
15	14891	15963	16135	16341	16477	16644	16809	16966	17137	17298	17460
16	15883	17027	17211	17430	17576	17753	17929	18097	18280	18451	18624
17	16876	17091	18287	18520	18674	18863	19050	19228	19422	19604	19788
18	17889	19156	19363	19609	19773	19972	20170	20360	20565	20757	20952
19	18861	20220	20438	20699	20871	21082	21291	21491	21707	21911	22116
20	19854	21284	21514	21788	21970	22192	22412	22622	22850	23064	23280
21	20847	22348	22590	22877	23063	23301	23532	23753	23992	24217	24444
22	21839	23412	23665	23967	24167	24411	24653	24884	25135	25376	25608
23	22832	24477	24741	25056	25265	25520	25773	26015	26277	26523	26772
24	23825	25541	25817	26146	26364	26630	26894	27146	27420	27677	27926
25	24818	26605	26892	27235	27462	27740	28015	28277	28562	28830	29100

§ 196. COUNTING BEATS OF SECONDS.

When the occurrence of a distant sound is not anticipated, we are unprepared to record the exact moment, and before an appropriate timekeeper can be procured an uncertain time has elapsed.

With some practice the beats of seconds can be counted in the mind with tolerable correctness without the aid of a timekeeper; which practice has been of great service to the author in astronomical observations. Practice to count seconds by the aid of an oscillating second pendulum, or by the second-hand on a watch, until the counting agree with the timekeeper, without attention to the pendulum or second-hand. With good practice the counting should not differ more than one second per minute.

When an unexpected distant sound is heard and its cause observed, we can always be ready to count seconds, and thus determine the distance.

In astronomical observations at sea it is customary to keep a watch in the hand, or to station an assistant at the chronometer to note the time when the observer says "stop;" but there are known cases when the captain has taken his observations without the aid of a watch or assistant, and walked slowly and comfortably to his cabin and noted the time of his observations from the chronometer, with no little amusement to other observers, who naturally supposed that the captain's observations could not be very correct, but to their surprise were found to be as correct as their observations with ordinary precautions. The captain counted in his mind the beats of seconds, and deducted the sum from the time observed on the chronometer.

The practice of counting seconds correctly is of great utility and service for estimating various movements. When the action is of very short duration, say less than 3 seconds, it is best to count half seconds, or even four times per second, and a short time may be determined with a correctness within a quarter of a second.

ASTRONOMY.

CREATION OF WORLDS.

§ 197. Matter in celestial space arranges itself into groups or nebulas by virtue of universal attraction, which by the aid of centrifugal force are finally divided into definite bodies, of which the largest occupies the centre around which the smaller revolve, and the group is called a planetary system.

Each fixed star is a central body of a planetary system like our sun.

The rotary motion of each group, nebula or of each body around its axis has been caused by collisions of the matters constituting that group or body.

The conditions under which a nebula can be formed into a planetary system are—*first*, that it must be set into a quick rotation; *secondly*, that it must consist of different kinds of matter; and *thirdly*, that its ingredients must be of such proportions as to admit of division by the forces of attraction and centrifugal.

Without the above conditions the nebula will remain permanent until it comes into collision with some other nebula or body.

It appears that each kind of matter is derived from different parts of space, and in its course of travel meets and mingles with other kinds of matter, whereby nebulas of a variety of shapes are formed. The different forms of nebulas are caused by different dynamical and chemical actions of the mingling matters. The act of collision of nebulas is distinctly seen through powerful telescopes, and the magnitude of the collision is so enormous that a change in form is hardly perceptible during a lifetime of observations. The forms of the different nebulas indicate their relative ages, which may be graduated from the form of a group of clouds to that of a permanently organized planetary system.

The majority of the well-defined nebulas located within our scope of observation are spherical or egg-shaped, with a bright central spot indicating the act of forming a sun, about which the surrounding matter gradually divides itself into separate masses, forming at length a planetary system.

Some nebulas are of the form of a ring, others consist of one or more spirals—a configuration which indicates the relative motions of

its constituent elements. We also find nebulas of the form of a spindle, having a bright body at each end.

The irregular nebulas, which are of the form of a group of clouds, may be classed as primary formations.

The period of time in which a primary nebula is thus formed into a planetary system may be many millions of years.

The operation of forming nebulas and the creation of worlds is going on all around us, even within our limit of observation through powerful telescopes, but the magnitude of that operation is too enormous for any human mind to conceive. The work is gradual and continued until all the matter in each part of space has assumed a definite form and motion. The matter in our part of space—that is, the space occupied by our planetary system and the neighboring ones—seems to be arranged into a definite form and motion; but in other parts of space, where groups and bodies are yet forming, some matter or portions of bodies may be led astray by repulsive force in collisions, and by being overtaken by superior force of attraction from other systems is drawn toward a central body, as is the case with matter constantly flowing into our sun.

When such stray matter is in the form of a solid body of perhaps thousands or millions of cubic miles in volume, it makes spots in the sun's photosphere; but a great deal of such flowing matter is in the form of a gas or powder (which we sometimes see as zodiacal light when it passes near to the earth) which makes no visible spots in the sun. Stray matter is often taken up by planets, as experienced on our earth, and we call it meteors.

Falling meteors change the motion of the earth and disturb our chronology, but their mass is so very small compared with that of the earth that it requires many years of observation for us to appreciate any such change; and as the meteors fall from all possible directions, some of them may counteract the action of their predecessors. The meteors which have fallen on the earth within our time of records and tradition have not changed our chronology more than, perhaps, a few seconds; but there has evidently been a time when very large bodies have struck the earth and changed its rotation both around its axis and around the sun, before which time the present location of the poles might have been at or near the equator.

The largest known meteor on our earth is lying on the pampa of Tucuman, near Otumpa, in the Republic of Argentine, South America, weighing about 16 tons.

A Comet is stray matter seeking a situation to supply other heavenly bodies with oxygen, hydrogen and nitrogen, and travels

from system to system, describing elliptical orbits around each central body, until its course is by chance directed near enough to some planet or sun to catch and retain the comet.

The return of a comet cannot be calculated or predicted except when its orbit is limited within our system. When the orbit extends outside of our system, the comet will likely never return, but is overtaken alternately by other planetary systems or groups of matter.

A planetary system may also have two suns, and subdivisions of groups or nebulas, in which the revolving bodies are called satellites, moving around a planet and forming a system within itself, but subject as one body to the main system. There are several planets in our system forming such a group, of which the earth and moon are one. The planets Saturn and Uranus have each eight satellites, Jupiter has four, and Neptune one.

The condition under which the revolving bodies are maintained in their regular orbits is, that the force of attraction of the central body is equal to the centrifugal force of the revolving one.

The orbits of periodical rotation are ellipses, in which the central body is in one of the foci. The orbit may accidentally be a circle, but there is no known planet or satellite which revolves in a perfect circle. A pendulum freely suspended and made to swing so as to describe a cone, the base of that cone would practically be an ellipse, for the reason that it is almost impossible to start the pendulum with such perfect velocity and direction in relation to its radius as to make it swing a perfect circle. Such is the case with the planetary orbits, in which the planets have not been started so as to describe perfect circles, and the orbits are also disturbed by the attraction between the planets.

§ 198. OUR PLANETARY SYSTEM.

It will be observed in the following tables that the other worlds in our system are of a different nature from that of our earth, and that no two of them are alike. Their difference of density is a striking feature; some of them are light as wood, others heavy as rock; and the planet Mercury must evidently be a chunk of precious metals. The density of the sun is given to be 1.128, compared with that of water, but the photosphere is included in the volume of that mean density. The core of the sun is evidently a very dense body. The column of density of the planets, however, indicates the tendency of the heavy materials to occupy the inner, and the lighter the outer, portion of a nebula or planetary system.

The following tables contain the principal elements of our planetary system:

Elements of the Planetary System.

The principal planets.	Signs.	Mean distance from the sun. Earth = 1	Miles.	Sidereal period of one revolution. Days.	Yrs. mts.	Rotat'n ar. axis. Hours.	Eccentric of orb. in part of. Sem. axis.
Sun . .	☉	. . .	. . . . .	. . . .	. . .	607.48	. . . .
Mercury	☿	0.3871	36,774,000	87.96926	0 2.93	24.05	0.2056179
Venus .	♀	0.7233	68,613,000	224.7008	0 7.49	23.21	0.0068334
Earth .	⊕	1.000000	95,000,000	365.25637	1 00	24.00	0.01677046
Moon .	☾	From ⊕	237,360	27.32116	0 0.91	0.0	0.0635
Mars .	♂	1.5236	144,742,000	686.97964	1 10.72	24.37	0.0932616
Jupiter .	♃	5.2028	494,266,000	4332.5848	11 10.49	9.56	0.0482388
Saturn .	♄	9.5388	906,186,000	10759.220	29 5.56	10.29	0.0559956
Uranus .	♅	19.182	1822,290,000	30686.820	84 3	9.30	0.0465775
Neptune	♆	30.037	2853,515,000	60126.72	165 7	. .	0.0087195

The principal planets.	Signs.	Diameter in miles.	Velocity in orbit miles per second.	Surface of planet. Earth = 1	Attraction on surface. Earth = 1	Light and ht. fr. sun. Earth = 1	Size of sun seen from planet.
Sun . .	☉	882000	. . . .	1243	28.3	. . . .	. . .
Mercury	☿	3140	30.4	0.144	0.51	6.673	2.584
Venus .	♀	7800	22.3	0.973	0.91	1.910	1.383
Earth .	⊕	7912	18.9	1.000	1.00	1.000	1.00
Moon .	☾	2160	6.35	0.0746	0.167	1.000	1.00
Mars .	♂	4100	11.33	0.269	0.500	0.430	0.656
Jupiter .	♃	87000	8.31	121	2.456	0.037	0.192
Saturn .	♄	79160	6.14	80	1.09	0.011	0.105
Uranus .	♅	34500	4.33	19	1.05	0.003	0.052
Neptune	♆	41500	3.45	27.6	1.10	0.001	0.033

The principal planets.	Signs.	Mass of planet. Earth = 1	Volume of planet. Earth = 1	Density of the planets. Earth = 1	Water = 1	Substances of equal gravity.	Inclination of orbit to ecliptic.
Sun . .	☉	355000	1378000	0.2543	1.128	Boxwood .	. . . . .
Mercury	☿	0.6966	0.06218	2.782	11.6	Lead . .	7° 0′ 8″.16
Venus .	♀	0.877	0.9531	0.9434	4.19	Topaz . .	3° 23′ 30″.75
Earth .	⊕	1.000	1.0000	1.0000	5.44	Iron pyrites	0 0 0
Moon .	☾	0.0114	0.02024	0.615	3.345	Limestone	5° 8′ 48″.
Mars .	♂	0.1313	0.1384	0.1293	0.575	Pine . .	1° 51′ 5″.08
Jupiter .	♃	317.5	1322.5	0.2589	1.15	Oak . .	1° 18′ 40″.31
Saturn .	♄	139.5	996.2	0.1016	0.451	Charcoal .	2° 29′ 28″.14
Uranus .	♅	198.	82.47	0.2796	1.24	Bitum. coal	0° 46′ 29″.91
Neptune	♆	20.	143.5	0.222	0.988	Camphor .	1° 46′ 58″.97

§ 199. INHABITATION OF WORLDS.

It would be unreasonable to suppose that our little earth is the only inhabited world amongst the millions of worlds in the universe, for wherever the proper proportions of the elements of life exist, organic bodies are formed in accordance with the conditions of operating elements.

Different organic bodies are composed of different material elements and ingredients in different proportions. The principal material elements required for the support of life are *oxygen*, *hydrogen*, *nitrogen* and *carbon*, and the physical elements are *force*, *motion* and *time*, which constitute the functions *light* and *heat*.

We know the operation, distribution and proportions of these elements on the earth's surface, but have no correct knowledge of the same in other worlds.

In regard to the other worlds in our system we know that the intensity of light on them is inversely as the square of their distance from the sun, but that does not give the heat and climate on those worlds. The heat which supports organic life on our earth is generated by the light passing through our atmosphere, and the denser the air is the more heat is generated, as proven by the decrease of temperature with the altitude above the level of the sea. The perpetual snow-line in the tropics is about 15,200 feet above the level of the sea, which proves a temperature of 32° Fahr. at that height.

Heat is generated by light passing through any transparent medium, and when the light is weak a denser medium is required for generating the heat necessary for the support of organic life.

As an illustration of the generation of heat by light, suppose a hollow cylinder or tube of say 3 feet long and 6 inches inside diameter, with a bottom in one end, made of a non-conducting substance for heat. Insert about forty pieces of plate-glass cut to fit the inside circle of the tube, which will be about three-fourths of an inch between each plate, leaving two or three inches between the last plate and the bottom. Place the tube so that the sun's light passes straight through it, and a heat will be generated at the bottom far above the temperature of boiling water.

The reason of this is, that light is power, which is composed of force and velocity; and when the velocity is reduced by the glass plates, the force, which is temperature, will be increased.

MOON.

The nearest world to our earth is the moon, which we know has little or no atmosphere, and is therefore not likely to be inhabited, although she receives the same intensity of light from the sun as does the earth. The topography of the moon's surface is very clear through powerful telescopes, but no sign of habitation has yet been discovered there. The planets are too remote for minute examination by the limited power of our present telescopes, but they are no doubt more or less inhabited.

MERCURY.

The planet *Mercury* is nearest to the sun, and receives over six times the intensity of light as does the earth, but with a light atmosphere, inhabitation is possible if the other elements necessary for the support of life exist there. *Mercury* is an irregularly shaped mass of precious metals, and cannot possess the abundance of the more useful elements composing our earth.

VENUS.

Venus is the second planet from the sun, and is nearly of the same size and composition as our earth, but the intensity of her light is about double that at the earth. The planet *Venus* is no doubt well inhabited. It is the nearest planet to us, but being inside of the earth's orbit, the sun's light interferes with our telescopic views of her topography.

EARTH.

The third planet from the sun is the *Earth*, upon which we live amidst its abounding glories established for us by the Creator of the Universe. Its population is about 1,400,000,000 inhabitants, which is the only known datum in the table of population of our planetary system.

We have also thousands of different species of animals, insects and plants to make up the inhabitation of our world.

MARS.

Mars is the fourth planet from the sun, and receives 0.43 the intensity of light as does the earth. The surface of this planet has a very conspicuous appearance, indicating land and seas, with a dense atmosphere in which we see floating clouds, and is probably inhabited with organizations suitable to the conditions of its elements.

Mars is the first planet upon which we expect to discover inhabitation when our telescopes are sufficiently advanced for that purpose.

ASTEROIDS.

Between the orbits of *Mars* and *Jupiter* are a number of planetoids, which are probably destined to be consolidated into one body with satellites. About 120 of them have been discovered and named, of which Ceres, Pallas, Juno, Vesta, Astræa, Hebe, Iris and Flora are the principal ones.

These planetoids are called *Asteroids*, and are probably not inhabited, on account of not having been long enough in a stable or permanent condition. Some low grade of organizations may exist on some of them.

JUPITER.

This is the largest planet in our system, and counting the asteroids as one, Jupiter is the sixth planet from the sun. The intensity of his light from the sun is only 0.037 of that of our earth, but he is surrounded with a dense atmosphere, in which can be seen floating clouds.

Jupiter has four satellites, forming a complete system within itself. The surface of *Jupiter* indicates the existence of land and seas well defined, and is probably inhabited.

SATURN.

Saturn is the seventh planet from the sun, and is surrounded with concentric rings, which appear in small telescopes to be only one ring. This planet has eight satellites, forming a system within itself.

The intensity of light in Saturn is only 0.011 of that on our earth. Saturn is too far from the earth for the limited power of our present telescopes to examine its surface, but is probably inhabited.

URANUS.

Uranus is the eighth planet from the sun, and is accompanied with a number of small satellites, of which eight have been discovered.

Satellites generally revolve around their planet in the same direction as the planet around the sun, and in orbits of a small inclination to the plane of the planet's orbit; but the satellites of Uranus revolve in an opposite direction, and with an inclination nearly at right angles to the plane of the planet's orbit.

NEPTUNE.

Neptune is the last known planet in our system, and was discovered by Leverrier in the year 1846; it is not visible to the naked eye, and can be observed only through a powerful telescope, by which one satellite has been found to accompany the planet.

SUMMARY OF INHABITATION AND CIVILIZATION IN WORLDS.

Inhabitants of other worlds are as comfortable with their combination of material and physical elements as we are with ours. They have different lengths of years, seasons, nights and days; different force of attraction, atmospheric pressure, light and heat, as shown in the table of elements of our planetary system. A body weighing one pound on the earth's surface weighs 2.456 pounds on *Jupiter*, and only half a pound on the planet *Mars*. The light on the surface of *Uranus* is only 0.003 of that on the Earth, but the optical organs of its inhabitants (if such exist) are constructed accordingly, so as to render them as comfortable with their light as we with ours.

Each variety of inhabitants is necessarily accommodated to the conditions of the operating elements, the most perfect organizations requiring more complicated elementary combinations.

We can justly claim that man is the most perfect organization known on our earth, but that claim cannot be extended to other worlds, particularly as long as we maintain armies and navies for offensive and defensive purposes; whilst the various forms of mischief, egoism and malignity which exist among men on earth, do not speak well for their civilization.

Considering the progress of man within the scope of history, and the fact that a large portion of the human race is yet in its primitive state, totally estranged from the surrounding progress, it appears that our earth cannot be very old in its present permanent condition. The idea of age in this connection comprehends millions of years.

We have good reason for supposing that organizations in other worlds outside of our system are far superior to our own, because the character of organizations does not depend only upon the existence of all the material and physical elements, but principally upon their proportions and distribution, which are evidently better classified in other worlds or in permanent nebulas existing millions of years before our earth was formed. The superior organizations in such old worlds have advantages not only in time and experience, but in greater varieties of physical phenomena upon which to exercise their knowledge and intelligence.

They may be sufficiently advanced in the science of optics to be able to extend their vision to our earth and examine our doings.

We can therefore be convinced that there exist in other worlds beings which are far superior to ourselves, whilst above all presides *the Creator of the Universe*, who superintends these myriad organizations, whose infinite inventions testify to His exhaustless and eternal power.

§ 200. LAW OF CELESTIAL MECHANICS.

The law of celestial mechanics was partly anticipated by several savans before and in Newton's time, but they did not succeed in arranging the physical elements so that the combinations would agree with their thoughts and observations.

Newton received a very valuable assistance from Kepler, as may be inferred from the correspondence between the two savans; which correspondence most likely led to the final establishment of the laws of celestial mechanics by Newton, which are as follows:

1st. *The areas described by the radius-vector of a planet is equal in equal times.*

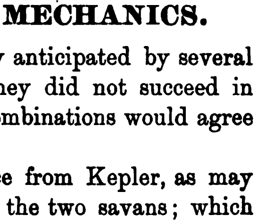
2d. *The planetary orbits are ellipses, in which the sun is in one of the foci.*

3d. *The force of attraction between any two masses is as the product of the masses and inversely as the square of their distance apart.*

4th. *The square of the times of one revolution of the planets are as the cubes of the semi-major axis of the orbits.*

In the accompanying illustration the ellipse m, a, b, c represents the orbit of a planet m revolving around the sun M. R = radius-vector, or distance between M and m. T = time of one revolution, and V = velocity of the planet in the orbit.

Fig. 225.



In accordance with the first law, the area described in a unit of time by the radius-vector, say from m to a, is equal to the area described in the same unit of time in any other part of the ellipse, say from b to c, or the areas of the sectors $m\ M\ a = b\ M\ c$.

The second law defines the orbit m, a, b, c to be an ellipse in which the sun M is in one of the foci.

The third law defines the force of attraction $F = \frac{M m}{R^2}$.

The centrifugal force of the planet m is $F = \frac{m V^2}{R}$.

Therefore $\frac{m V^2}{R} = \frac{M m}{R^2}$, and $V^2 = \frac{M}{R}$, or $V = \sqrt{\frac{M}{R}}$.

As the mass M of the sun is constant for all his planets, it follows that the velocity V in the orbit is inversely as the square root of the radius-vector.

The semi-major axis $o\,p$ is a function of the time T of one revolution of the planet in its orbit, but $o\,p$ is a function of the radius-vector R, for which reason we can place

$$R = V\,T, \quad \text{of which} \quad V = \frac{R}{T}, \quad \text{and} \quad V^2 = \frac{R^2}{T^2} = \frac{M}{R}.$$

The mass M is constant for all the planets, and we have

$T^2 = R^3$, the fourth law.

The square of the times of one revolution are as the cubes of the radius-vector.

§ 201. TO FIND THE MASS OF THE SUN.

The centrifugal force of the earth revolving around the sun is equal to the force of attraction between the two bodies.

M = mass of the sun, and m = mass of the earth.

D = distance in feet, R = distance in miles, from the earth to the sun.

n = revolutions per minute of the earth around the sun.

Centrifugal force $$F = \frac{m\,V^2}{D}.$$

Velocity $$V = \frac{2\,\pi\,D\,n}{60}, \quad \text{and} \quad V^2 = \left(\frac{2\,\pi\,D\,n}{60}\right)^2.$$

Centrifugal force $$F = \frac{m(2\,\pi\,D\,n)^2}{D\,60^2} = m\,D\left(\frac{2\,\pi\,n}{60}\right)^2.$$

Centrifugal force, $$m\,D\left(\frac{2\,\pi\,n}{60}\right)^2 = \frac{M\,m}{28693080\,D^2}, \text{ force of attraction.}$$

Mass of the sun, $$M = 28693080\,D^3\left(\frac{2\,\pi\,n}{60}\right)^2, \text{ in matts.}$$

The sidereal number of revolutions per minute of the earth around the sun will be

$$n = \frac{1}{60 \times 24 \times 365.25} = \frac{1}{511350}.$$

*log.*5.7087182.

$$n^2 = \frac{1}{259679000000}.$$

*log.*11.4174364.

$$28693080\left(\frac{2\,\pi\,n}{60}\right)^2 = \frac{1}{831002}.$$

*log.*5.9190422.

$D = 5280\ R = 5280 \times 95{,}000{,}000 = 501{,}600{,}000{,}000$ feet.

$D^3 = 126{,}203{,}844{,}096{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$ cubic feet.

From paragraph 51 we know that the mass of the earth is

402,735,000,000,000,000,000,000 matts. *lon.*23.6050086.

Then the mass of the sun compared with that of the earth will be

$$M = \frac{126{,}203{,}844{,}096{,}000{,}000}{402{,}735 \times 831002} = 377100.$$

*log.*5.5764616.

This result is a little higher than that in the table, page 284.

§ 202. TO FIND THE DISTANCE FROM THE SUN TO ANY OF HIS PLANETS.

Knowing that the centrifugal force of any planet revolving around the sun is

$$F = m\, D\left(\frac{2\,\pi\,n}{60}\right)^2,$$

and that this centrifugal force is equal to the force of attraction,

$$F = \frac{M\,m}{\varphi\, D^2},$$

we have
$$\frac{M\,m}{\varphi\, D^2} = m\, D\left(\frac{2\,\pi\,n}{60}\right)^2.$$

$$\varphi\, D^3\left(\frac{2\,\pi\,n}{60}\right)^2 = M, \quad \text{and} \quad D^3 = \frac{M}{\varphi}\left(\frac{60}{2\,\pi\,n}\right)^2.$$

$$\text{Distance} \quad D = \sqrt[3]{\frac{M}{\varphi}\left(\frac{60}{2\,\pi\,n}\right)^2}.$$

Let t denote the time in minutes of one sidereal revolution of any planet around the sun.

Then $t = \frac{1}{n}$, and $n = \frac{1}{t}$, which inserted in the formula for D, we have

$$D = \sqrt[3]{\frac{M}{\varphi}\left(\frac{60\,t}{2\,\pi}\right)^2}$$

Let T denote the time in days of one sidereal revolution of a planet around the sun, and we have

$$t = T \times 60 \times 24 = 1440\ T.$$

$$D = \sqrt[3]{\frac{M}{\varphi}\left(\frac{86400\ T}{2\pi}\right)^2}.$$

In this formula we have given the masses of the sun:

Log. mass of the earth		23.6050086
Log. earth mass of sun	add.	5.5800176
		29.1850262
Log. φ	sub.	7.4577772
Log. $\dfrac{M}{\varphi}$		21.7272490
Log. $\left(\dfrac{86400}{6.28}\right)^2$	add.	8.2771082
Divide by 3		30.0043572
Log. 10033500000 = ...		10.0014524

$$D = 10033500000\sqrt[3]{T.}$$

Let R denote the radius of the orbit of a planet in statute miles.

$$D = 5280\ R, \quad R = \frac{D}{5280}.$$

$$\frac{1033500000}{5280} = 1900290. \qquad log. 6.2788185.$$

$$\text{Radius-vector} \quad R = 1900290\sqrt[3]{T^2}$$

This formula gives the distance from the sun to any of his planets, when the time T in solar days of one sidereal revolution is known.

Example. It is known that the planet Jupiter makes one revolution around the sun in a time $T = 4332.6$ days. Required his mean distance from the sun?

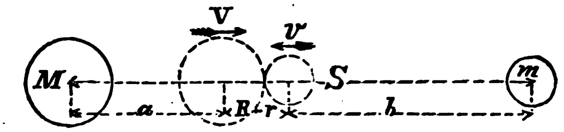
$$R = 1900290\sqrt[3]{4332.6^2} = 505030000 \text{ miles.}$$

There is one item omitted in the sections 201 and 202—namely, that the planets do not revolve strictly around the sun's centre, but around the common centre of gravity of the two bodies. The whole planetary system revolves around its common centre of gravity.

§ 203. TO FIND THE TIME IN WHICH TWO BODIES WOULD BE DRAWN TOGETHER BY THE FORCE OF THEIR OWN ATTRACTION.

Assume two bodies M and m to be held at a distance S apart; when let loose or free to move, their force of attraction will draw them together.

Fig. 226.



Let M and m denote the respective masses, expressed in matts.

S = distance apart in feet.

T = time in seconds in which they would be drawn together.

V and v = velocities in feet per second.

R and r = the respective radii of the masses in feet, supposing them to be spherical.

a and b = the respective distances in feet moved by the masses to the point of collision.

φ = 28693080, the coefficient of attraction.

As the force of attraction is the same on each body, we have

$$a : b = m : M, \quad \frac{m}{M} = \frac{a}{b}, \quad \text{or} \quad a = \frac{m\,b}{M}.$$

$$S = a + b + R + r, \quad b = S - R - r - \frac{m\,b}{M} = \frac{S - R - r}{(1 + \frac{m}{M})}.$$

$$F : m = v : t, \quad \text{of which} \quad v = \frac{F\,t}{m}.$$

$$F = \frac{M\,m}{\varphi(S - b - \frac{b\,m}{M})^2}.$$

$$v = \frac{M\,m\,t}{m\,\varphi[S - b(1 - \frac{m}{M})]^2} = \frac{\partial b}{\partial t}.$$

$$M\,t\,\partial t = \varphi[S - b(1 - \frac{m}{M})]^2\,\partial b.$$

$$\int t\,\partial t = \frac{\varphi}{M}\int S^2\,\partial b - 2\,S(1 - \frac{m}{M})b\,\partial b + (1 - \frac{m}{M})^2 b^2\,\partial b.$$

$$t = \sqrt{2\frac{\varphi\,b}{M}[S^2 - S b(1 - \frac{m}{M}) + \frac{b^2}{3}(1 - \frac{m}{M})^2]}. \quad . \quad . \quad 1$$

Example. Suppose the moon to be stopped in her circular motion around the earth; the force of attraction would then draw the two bodies together.

It is required to know the time in which the moon would fall to the earth?

The masses of the earth and moon are,

$M = 402,735,000,000,000,000,000,000$ matts...*log.* 23.6050194

$m = 4,591,000,000,000,000,000,000$..............*log.* 21.6619135

$S = 1253260800$ feet................................*log.* 9.0980413

$R = 20887680$ feet, and $r = 5702400$ feet.

$b = 121282000$ feet................................*log.* 8.0838052

$\varphi = 28693080$...*log.* 7.4577772

$\frac{m}{M} = 0.0114, \quad \left(1 - \frac{m}{M}\right) = 0.9886$.............*log.* $0.9941411 - \dot{1}$

Insert these values in Formula 1, and the result will be 171180 seconds, which is 1 day, 23 hours and 33 minutes, the time required for the moon to fall upon the earth.

A very small body m falling from a great distance S into a very large body M, makes $\left(1 - \frac{m}{M}\right) = 1$, and $b = S$, nearly; and the time of fall may be estimated by

$$t = S\sqrt{\frac{\varphi S}{3 M}}.$$

The bodies M and m are moved in opposite directions with their common force of attraction; the velocities V and v will consequently be

$$V : v = m : M, \quad \text{and} \quad M V = m v.$$

The momentums being alike, the masses will stop one another's motion.

Supposing the bodies to have no elasticity and to be crushed into one another in the collision, the work of that collision will be the sum of the works stored in each body.

The work in M is $K = \frac{M V^2}{2}$.

The work in m is $k = \frac{m v^2}{2}$.

Work of collision $K + k = \frac{M V^2 + m v^2}{2}$.

The consolidated masses will be brought to rest after the work of collision.

$$K : k = M\,V^2 : m\,v^2 = m : M = V : v.$$

The work stored in each mass is inversely as the masses and velocities.

$$\delta K = F\delta a = \frac{M\,m\,\delta a}{\varphi\,[\,S + a(1+\frac{M}{m})]^2}.$$

$$\delta k = F\delta b = \frac{M\,m\,\delta b}{\varphi\,[\,S - b(1-\frac{m}{M})]^2}.$$

The force of attraction between the earth and moon is

$$F = \frac{M\,m}{\varphi\,S^2} = \frac{402{,}735\text{etc.} \times 4{,}591\text{etc.}}{28693080 \times 1253260800^2} = 410272000000000000000 \text{ pounds.}$$

If this force was constant and applied on the moon in the direction of the earth, it would bring the two bodies together in a time

$$T = \sqrt{\frac{2\,M\,S}{F}} = \sqrt{\frac{2 \times 4591\text{etc.} \times 1253\text{etc.}}{410272\text{etc.}}} = 529610 \text{ seconds,}$$

or 6 days, 3 hours, 6 minutes and 45 seconds.

§ 204. MAXIMUM ATTRACTION.

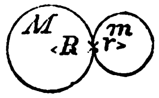
The force of attraction between any two bodies M and m at a distance S apart is

$$F = \frac{M\,m}{\varphi\,S^2}.$$

The force of attraction is inversely as the square of the distance, consequently, when the distance is infinitely small, the attraction will be infinitely great; but there is a limit to the distance between the centres of attraction—namely, when the bodies are in close contact their centres of attraction are still at a considerable distance apart.

Let two spherical masses M and m, and of radii R and r, be in close contact to one another; their maximum attraction will then be

Fig. 227.



$$F = \frac{M\,m}{(R+r)^2}.$$

If the masses are forced into one another, the force of attraction will be diminished.

The masses of spheres of equal substance is as the cube of the radii, and the force of attraction can be expressed by

$$F = \frac{R^3 r^3}{(R+r)^2}.$$

When the two spheres are of equal radii, or $R = r$, the force of attraction is

$$F = \frac{R^3 R^3}{4 R^2} = \frac{R^4}{4}.$$

$D = 2R$, the diameters of the sphere, and the force of maximum attraction is a limited function $F = D^4$.

That is to say, the force of attraction between equal spheres in contact is as the fourth power of the diameter, or the force of attraction increases directly as the fourth power of the distance between the centres of attraction.

§ 205. ELEMENTS OF THE EARTH AND MOON.

The earth and the moon revolve around their common centre of gravity in periods of 29.53 days from new moon to new moon.

M = mass of the earth, and m = that of the moon.

r = radius of the earth, and R = distance from the centre of the earth to the centre of the moon. $R = 60\,r$.

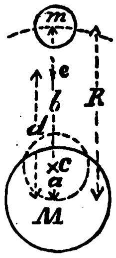
Let c denote the position of the common centre of gravity of the two bodies, a = distance from c to the centre of the earth, and b = distance from c to the centre of the moon.

Let d denote the position of the centre of attraction of the earth and moon. It is required to find the centres c and d?

Centre of Gravity of the Earth and Moon.

In regard to the centre of gravity c, we have the static momentums

Fig. 228.



$$M a = m b, \qquad \text{but } R = a + b = 60\,r;$$

$$b = 60\,r - a;$$

$$a = \frac{m b}{M} = \frac{m(60\,r - a)}{M}.$$

$$M a = m\,60\,r - m\,a;$$

$$M a + m a = 60\,m\,r, \quad \text{or} \quad a(M+m) = 60\,m\,r.$$

$$a = \frac{60\,m\,r}{M+m}.$$

Assume the radius of the earth $r = 1$.

$$
\begin{aligned}
Log.60 &= 1.7781513 \\
log.m &= \underline{21.6619135} \\
log.60M &= 23.4400648 + \\
log.M+m &= \underline{23.6050086} - \\
log.0.684 &= 9.8350562
\end{aligned}
$$

That is, the distance of the centre of rotation of the earth and moon is at 0.684 r from the centre of the earth, or 0.316 of the earth's radius under the surface of the earth where the moon is in the zenith.

The centres of the earth and moon thus describe elliptic orbits around the centre c as the common focus, whilst the centre c moves in an elliptic orbit around the sun.

The radius of the earth is about 3956 miles, or 20887680 feet.

$$a = 3956 \times 0.684 = 2705.904 \text{ miles.}$$
$$b = 3956 \times 60 - 2705.904 = 157534.096 \text{ miles.}$$

The velocity of a body moving in a circle is $\dfrac{2\pi R}{T}$.

The earth and moon make one sidereal revolution in 27.32 solar days. $27.32 \times 24 = 655.68$ hours.

Velocity, $$= \frac{2 \times 3.14 R}{655.68} = \frac{R}{104.407}.$$

Earth's centre, $$v = \frac{2705.904}{104.407} = 25.917 \text{ miles per hour.}$$

Moon's centre, $$V = \frac{157534.096}{104.407} = 1508.8 \text{ miles per hour.}$$

Centre of Attraction Between the Earth and Moon.

In regard to the centre of attraction between the earth and moon, we have

Force attraction, $$F = \frac{Mm}{R^2}.$$

It is required to find a point in the straight line between the earth and the moon where a body would be equally attracted from the two bodies, which is when

$$\frac{M}{d^2} = \frac{m}{(R-d)^2}, \quad \text{of which } d = \frac{R}{M-m}(M - \sqrt{Mm}).$$

Assuming the mass of the moon as unit, or $m=1$, then the mass of the earth will be $M=87.7$. When $R=60$ the distance d will be expressed in radii of the earth.

$$d=\frac{60}{87.7-1}(87.7-\sqrt{87.7\times 1})=54.2.$$

$d=54.2\times 3956=214415$ miles from the centre of the earth to the centre of attraction. A body placed at e, Fig. 228, would be equally attracted from the earth and moon.

§ 206. ORBITS OF THE EARTH AND MOON.

Fig. 229 represents the orbits of the earth and moon moving in the direction of the arrow, as seen from the North Star. The dotted line a represents the orbit described by the common centre of gravity of the earth and moon, which is an ellipse in which the sun is in one of the focii. The eccentricity of the ellipse is only 0.0168 of the major axis.

Fig. 229.

The drawn line b represents the orbit described by the centre of the moon, and the line c that of the earth. The line 0, 1, 2, 3 and 8 represents the radius-vectors from the sun. The illustration represents the orbits for one lunar month, or from new moon to new moon, which are accomplished in a time of 29 days, 31 minutes and 48 seconds.

The mean velocity of the common centre of gravity in the dotted orbit a is 68,091 miles per hour.

The following table shows the mean sidereal velocities in miles per hour of the earth and moon in their respective orbits at different positions of the moon.

Sidereal Velocities of the Earth and Moon.

No.	Positions.	Velocities in miles per hour. Earth.	Moon.
0	New moon	68086	66582
1	Middle of first quarter	68110	67024
2	Half moon	68091	68091
3	Middle of second quarter	68072	69158
4	Full moon	68069	69600
5	Middle of third quarter	68072	69158
6	Half moon	68091	68091
7	Middle of fourth quarter	68110	67024
8	New moon	68086	66582

The greatest difference of velocity of the moon is 3013 miles per hour, a distance of about that between New York and Liverpool.

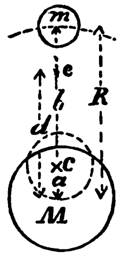
The moon travels a distance of about 10 times the diameter of the earth per hour.

A point on the equator describes a worm-line on the earth's orbit.

§ 207. TO FIND THE MASS OF THE EARTH AND MOON.

The notation of letters will be the same as in the preceding paragraph.

Fig. 230.



The earth M and moon m revolve around their common centre of gravity c. Having given the earth's radius and the moon's horizontal parallax, we obtain the distance $R = 237360$ miles, or $R = 1253260800$ feet, between the centres of the two bodies.

The force of attraction between the two bodies is

$$F = \frac{Mm}{\varphi R^2}, \quad \text{in which } \varphi = 28693080.$$

The centrifugal forces are

$$F = \overset{\text{Earth.}}{Ma\left(\frac{2\pi}{60\,T}\right)^2} = \overset{\text{Moon.}}{m\,b\left(\frac{2\pi}{60\,T}\right)^2}.$$

T = time in minutes of one sidereal revolution of the system, which is

$$27^d,\ 7^h,\ 43^m,\ 11^s, \quad \text{or} \quad T = 39343.183 \text{ minutes.}$$

$$\text{Call } \gamma = \left(\frac{2\pi}{60\,T}\right) = \left(\frac{2 \times 3.14}{60 \times 39343.183}\right)^2 = \frac{1}{141150000000}.$$

*log.*11.1496796.

The centrifugal forces are equal to the force of attraction, or

$$\frac{Mm}{\varphi R^2} = M a \,\mho, \qquad \text{and } \frac{Mm}{\varphi R^2} = m b \,\mho.$$

$$\frac{m}{\varphi R^2} = a \,\mho, \qquad \text{and } \frac{M}{\varphi R^2} = b \,\mho.$$

$$\text{Call} \quad Z = \varphi \,\mho = \frac{28693080}{141150000000} = \frac{1}{4919.26}.$$

*log.*3.691919024.

$$m = R^2 Z a, \qquad \text{and } M = R^2 Z b.$$

$$R = a + b.$$

$$M + m = R^2 Z a + R^2 Z b = R^2 Z(a + b) = R^3 Z.$$

The mass of the earth and moon $M + m = R^3 Z$.

$$M + m = \frac{1253260800^3}{4919.26}.$$

$M + m = 402,820,000,000,000,000,000,000$ matts.

This calculation gives the mass of the earth and moon a little less than that on page 294, owing to the data not being very correct.

The masses of the earth and moon cannot be calculated separately from the preceding formulas, because the given data R and T are constant for any proportion of M and m. If $M = m$, the centre c would be in the middle between the earth and moon, but R and T would be unaltered, and the force of attraction and centrifugal forces would still be alike.

§ 208. TO FIND THE MASSES OF ANY TWO HEAVENLY BODIES REVOLVING AROUND ONE ANOTHER.

Let S denote the distance in statute miles between the centres of the two bodies, and t = time in days of one sidereal revolution, M and m being the masses of the revolving bodies expressed in units of that of the earth and moon.

$$M + m = \frac{S^3}{17914680000000\, t^2}.$$

*log.*13.2532071.

Required the mass of the earth and the sun?

$$S = 95{,}000{,}000 \text{ miles}, \qquad t = 365.242 \text{ days}.$$

$$M + m = \frac{95000000^3}{17914680000000 \times 365.25637} = 358740.5.$$

When the mass of the earth is 1, that of the sun will be 358739.5.
When the mass of one of the bodies is known, that of the other is

$$m = \frac{S^3}{17914680000000\, t^2} - M.$$

The mean distance of the planet Jupiter from the sun is estimated to be $S = 494265000$ miles, and the planet makes one sidereal revolution in $t = 4332.6$ days. Required the mass of Jupiter?

$$M + m = \frac{494265000^3}{17914680000000 \times 4332.6^2} = 359067.$$

Log.S =	8.6939599	*log.*t =	3.6367486
mult. cube	3	mult. square	2
	26.0818797		7.2734972
subt.	20.5267043	*log.*179, etc. =	13.2532071
*log.*359067 =	5.5551754		20.5267043

The masses	$M + m = 359067$
Mass of the sun	$M = 358739$
Mass of Jupiter	$m = \quad 328$

This includes also the mass of Jupiter's satellites.

The mass can thus be calculated of any planet whose distance from the sun and time of sidereal revolution is known.

When the sum of the masses of two revolving bodies is known, their distances apart S in miles and time t in days of one sidereal revolution will be as follows:

Distance $S = 26165.9\sqrt[3]{t^2(M+m)}$ in ~~minutes~~ miles.

Time $t = 4232563\sqrt{\frac{M+m}{S^3}}$ in days.

Volume and Density.

For the volume and density of the heavenly bodies it is necessary to know their mass and diameter.

D = diameter of the planet in statute miles.
P = volume in cubic miles.
Q = density compared with water at 39° Fahr.
M = mass of the planet compared with that of the earth.

$$\text{Volume} \quad P = \frac{\pi D^3}{6}.$$

$$\text{Density} \quad Q = \frac{1400350000000\, M}{P}.$$

Example. Required the volume and density of the earth?

$D = 7912$ miles, the diameter of the earth.

$$\text{Volume} \quad P = \frac{3.14 \times 7912^3}{6} = 259014000000 \text{ cubic miles.}$$

$$Q = \frac{1400350000000}{259014000000} = 5.4025.$$

That is to say, the earth is 5.4025 times heavier than an equal volume of water.

The interior of the earth is probably composed of metallic sulphurets, principally iron and copper. The specific gravity of these substances is about 5.4.

The specific gravity of the planet Mercury is variously stated between 6 and 15 times that of water; it must evidently be a purely metallic body.

The volume and density of the planet Jupiter will be, when $D = 87000$, and $M = 328$.

$$\text{Volume} \quad P = \frac{3.14 \times 87000^3}{6} = 336{,}770{,}000{,}000{,}000 \text{ cubic miles.}$$

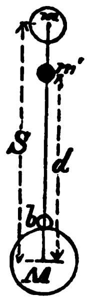
$$\text{Density} \quad Q = \frac{1400350000000 \times 328}{336770000000000} = 1.36389.$$

The density of Jupiter is 1.36389 compared with that of water, or 0.252 of that of the earth.

The astronomical data appear to be yet very incorrect, and it is therefore difficult to make them agree with the physical laws involved.

§ 209. TO FIND THE VELOCITY WITH WHICH A BODY SHOULD be started from the earth's surface in order to reach the Moon.

Fig. 231.



In the accompanying illustration M represents the earth and m the moon. The point m' is the centre of attraction between the two bodies. A body at m' would be equally attracted by M and m; therefore if the body m' is overtaken by the earth's attraction, it would arrive at b with a velocity equal to that with which it should be started from the earth's surface in order to reach the moon.

M, m, and m' denote the masses of the respective bodies expressed in matts.

S = distance in feet between the centres of the earth and moon.
d = distance in feet of the body m' from the centre of the earth.
F = force of attraction of the earth on the body m'.
f = force of attraction of the moon.

$$F = \frac{M\,m'}{\varphi\,d^2}, \text{ and } f = \frac{m\,m'}{\varphi(S-d)^2}.$$

The force acting on the body m' in its fall to the earth will be

$$F - f = \frac{M m'}{\varphi\,d^2} - \frac{m\,m'}{\varphi(S-d)^2}.$$

$$F - f = \frac{m'}{\varphi}\left(\frac{M}{d^2} - \frac{m}{(S-d)^2}\right).$$

Let v denote the velocity of the fall in feet per second.

T = time of fall in seconds.

$$(F-f) : M = v : T, \text{ of which } (F-f) = \frac{Mv}{T}.$$

$$\frac{m'\,v}{T} = \frac{m'}{\varphi}\left(\frac{M}{d^2} - \frac{m}{(S-d)^2}\right).$$

$$v = \frac{T}{\varphi}\left(\frac{M}{d^2} - \frac{m}{(S-d)^2}\right). \quad . \quad . \quad . \quad . \quad 1$$

$T = \frac{\partial d}{\partial v}$, when d and v are variables, which inserted in Formula 1 will be

$$v = \frac{\partial d}{\varphi\, \partial v}\left(\frac{M}{d^2} - \frac{m}{(S-d)^2}\right).$$

$$\int v\, \partial v = \frac{1}{\varphi}\int\left(\frac{M\, \partial a}{d^2} - \frac{m\, \partial a}{(S-d)^2}\right).$$

$$\frac{v^2}{2} = \frac{1}{\varphi}\left(\frac{M}{d} - \frac{m}{(S-d)}\right) + C.$$

The operation is to be integrated from the centre of attraction to the surface of the earth, where $d = R$ the radius of the earth, and when $v = 0$ we have

$$C = -\frac{1}{\varphi}\left(\frac{M}{d} - \frac{m}{(S-d)}\right).$$

Then $$V = \sqrt{\frac{2}{\varphi}\left(\frac{M}{R} - \frac{m}{(S-R)} - \frac{M}{d} + \frac{m}{(S-d)}\right)}.$$

$$V = \sqrt{\frac{2(d-R)}{\varphi}\left(\frac{M}{R\, d} + \frac{m}{(S-d)(S-R)}\right)}.$$

We have all the data given in this formula—namely,

$M = 402,735,000,000,000,000,000,000$ matts	23.6050194
$m = 4,591,065,000,000,000,000,000$ matts	21.6619135
$d = 214415$ miles $= 1132111200$ feet	9.0538890
$S = 1253260800$ feet	9.0980413
$R = 20887680$ feet	7.3198903
$\varphi = 28693080$	7.4577772

The body m' will then fall on the earth with a velocity $V = 40781$ feet per second.

That is to say, a body started with a velocity of 40781 + feet per second from the surface of the earth in the direction of the moon would reach the centre of attraction and fall into the moon. The velocity with which it would arrive on the moon's surface is calculated by the same formula, for which

	Logarithms.
$R = 5702400$ feet, the radius of the moon	6.7560577
$d = 121149600$ feet	8.0833220
$S - R = 1247558400$ feet	9.0960611
$S - d = 1132111200$ feet	9.0538890
$d - R = 115447200$ feet	8.0623836

In this case M = mass of the moon, and m = mass of the earth.

The velocity with which a body would fall from the centre of attraction into the moon would be only 7468 feet per second. A body thrown from the moon with a velocity of 7468 feet per second towards the earth would reach the earth with a velocity of 40781 feet per second.

The time required for a body to fall from the centre of attraction to the surface of the earth is obtained by solving the Formula 1, and placing $V=\frac{\partial d}{\partial T}$.

$$\frac{\partial d}{\partial T}=\frac{T}{\varphi}\left(\frac{M}{d^2}-\frac{m}{(S-d)^2}\right).$$

Find the value of T in this formula, which will give the required time in seconds.

APPENDIX.

DUODENAL SYSTEM OF ARITHMETIC, MEASURES, WEIGHTS AND COINS.

THE object of appending a treatise on a new system of arithmetic and metrology is to demonstrate what can be done with that subject, which demonstration might by that means be conveniently accessible to the student and to the public.

The problem of an international and complete system of metrology has at all times been esteemed an important desideratum, but no attempt has yet been made to remove the principal difficulty which is in the way, and we can expect no satisfactory metrology until its primary obstacle is removed.

The base **ten**, which is adopted in our present arithmetic, does not admit of binary and trinary divisions, as required in metrology. This is the principal difficulty in the way of establishing a satisfactory system of measures, weights and coins.

The number *10* is actually the worst even number that could have been selected as a base of numeration, for which either *8*, *12*, or *16* would have been better.

The inconveniences of the decimal base in metrology are well known, and have been explained at various times by various writers; but the present arithmetic is so thoroughly incorporated with civilization that it appears difficult to unlearn and get rid of the same for the substitution of something better.

The American Pharmaceutical Association appointed a committee, of which Alfred B. Taylor of Philadelphia was chairman, for the purpose of investigating the present condition of metrology with a view to its improvement, who gave the subject a very careful and deliberate consideration.

An elaborate report containing over *100* octavo pages of fine print was prepared and read before the annual session of the Association, held in Boston September *15*, *1859*. This report explains the inconveniences of the decimal arithmetic and of the French metrical system, illustrated by quotations from various authors of high authority.

In the course of this report Mr. Taylor proposed and elucidated an **Octonal System** of arithmetic and metrology.

Octonal System.

The octonal system has *8* to the base, which admits of binary division to unity without fractions. It would be an easy system to learn and manage in both arithmetical and mental calculations, but it requires a greater number of figures than the decimal system in expressing high numbers, and eight is too small as a base.

The octonal system, moreover, does not admit of trinary division, as is required in the circle and time.

Decimal System.

The decimal arithmetic is of Hindoo origin, and was imported into Arabia some one thousand years ago, from which it was spread throughout Europe and the entire civilized world.

The base **ten** originated from the *10* fingers, which were used for counting before characters were formed to denote numbers.

The base *10* admits of only one binary division, which gives the prime number *5* without fraction. The trinary divisions give an endless number of decimals. The decimal system is therefore not well suited for metrology, in which binary and trinary divisions are required.

It is this defect of the decimal system which has caused confusion in metrology and discordance among nations respecting the adoption of one common system of measures; which problem will never be satisfactorily solved as long as decimal arithmetic is maintained.

By examining the tables of measures and weights of different nations we find that binary and trinary divisions are invariably preferred, notwithstanding that decimal arithmetic must be used in their calculation.

The French decimal metrology is perhaps the best that can be devised in connection with decimal arithmetic; it looks very inviting and simple on paper, but what is gained by the metrical system in calculations is lost in the shop and market.

The defects of the metrical system are the defects of our arithmetic itself, and as long as decimal arithmetic is maintained the French system is the best of all that have yet been devised.

The slow adoption of the metrical system by other nations is sustained by good reasons—namely, that it does not constitute a complete, uniform and convenient system of metrology. The decimal system, as before stated, is not applicable to the admeasurement of the circle, of time and of the compass, where binary divisions are indis-

pensable. The circle requires both binary and trinary divisions, neither of which can be accommodated by the decimal base.

When the metrical system was first established in France, it was intended to decimate also the circle and the time, which was soon found to be impractical and the idea abandoned.

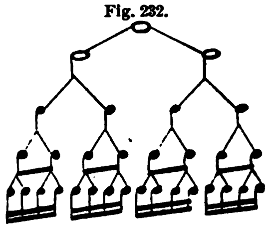
The French metrology is therefore not a complete system, and it has been renounced for all measures in astronomy, geography, navigation, time, the circle and the sphere, where it is inapplicable.

The decimal system is also inapplicable in music, where the binary and trinary divisions are invariably used.

Music represents the natural disposition of the mind to arrange or classify quantities. The musical bar is divided into *halves*, *quarters*, *eighths* and *sixteenths;* and also into *thirds*, *sixths*, *ninths* and *twelfths;* but we never find music divided into tenths.

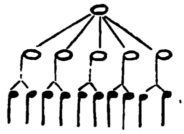
The most natural or binary division of music is represented thus:

Fig. 232.



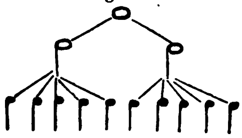
A bar of music divided by the decimal system would appear thus:

Fig. 233.



or if you please

Fig. 234.



No music could be produced by either of these last divisions, but a mechanical noise only could be made by it.

The lowest grade of man, and even animals, sing binary music. Even an Australian magpie can be taught to whistle any ordinary song as correctly as played on a musical instrument; whereas a decimal division of music could never be learned and appreciated even by the highest intelligence.

Such is also the comparison between binary and decimal arithmetic. Decimal arithmetic is a heavy burden upon the mind, and limits mental calculations within a very narrow compass; whilst binary or trinary arithmetic would become natural to the mind like music, and render mental calculations as easy as music played by the ear.

The Folded French Metre.

The French metre is difficult to fold into a convenient shape for the pocket. The ten-folded metre with lap-joints is a very convenient form for approximate measurements, but cannot be relied upon for correctness, because the numerous lap-joints cannot be made permanently accurate, and moreover the lap-joints do not form a straight but a broken line. The metre folded into five parts with lap-joints is an odd affair.

The two-folded metre of five decimetres in each part, of about *20* inches long, is too large for the pocket.

The four-folded metre makes two and a half decimetres in each part of about *10* inches long, which will answer for the pocket; and is perhaps the best form of the French metre when made with regular hinges like the English four-folded rule, but it is still a broken measure.

An international association for obtaining a uniform decimal system of weights, measures and coins has been in existence for over thirty years, and has yet accomplished very little. The object of this association is wholly for the introduction of the French metrical system, which has met with the most natural and reasonable objections—namely, that it is not a complete system, and that it is inconvenient in the shop and in the market; but the strong influence of this association has induced many governments to force that system upon their people.

In practice, we want our units divided into the simplest and most natural fractions—namely, *halves*, *thirds*, *quarters*, *sixths*, *eighths*, etc.—which cannot be done by the metrical system, or decimal arithmetic without long tails of figures commencing with 0.

For instance, the simple fraction $\frac{1}{3}$ expressed by decimals is 0.33333....... without end, and will never be correct, and requires a good education to understand the true meaning of it. The good scholar manages the decimal fractions as easily as a musician plays on his hand-organ, but the fraction 0.33333 is not so easily understood by the majority of the people, who will naturally ask what it means. In the answer it is necessary to explain that the unit is divided into 100000 parts, and 33333 of those parts is nearly $\frac{1}{3}$ of the whole. The people will then surely reply that they are not willing to cut their things up into 100000 parts and lose a portion by the division in order to get it into three.

Duodenal System.

Charles XII. of Sweden proposed to introduce a duodenal system of arithmetic and metrology. The king complained of ten as a base, and said, "It can be divided only once by 2, and then stops." The number *12* can be divided by 2, 3, 4 and 6 without leaving fractions; and divided by 8 gives $\frac{2}{3}$, by 9 gives $\frac{3}{4}$, and by 10 gives $\frac{5}{6}$, all convenient fractions for calculation.

The number *12* has always been a favorite base in metrology.

The old French foot was divided into *12* inches, the inch into *12* lines, and the line into *12* points. The *dozen* is a well-known base adopted all over the world; *12 dozens* is a *gross*, and *12 gross* is a *great-gross*. We have *12* months in a year, *12* hours in a day, *12* signs in the zodiac, *12* musical notes in an octave. The old Roman metrology was based on *12*, like the English foot and the Troy pound.

A writer in the Edinburgh Review (Jan., *1807*, vol. *9*, page *376*) regrets that the philosophers of France, when engaged in making so radical a change in the measures and standards of the nation, did not attempt a reform in the popular *arithmetic*. He, being in favor of a duodenal system, says, "The property of the number *12* which recommends it so strongly for the purpose we are now considering is its divisibility into so many more aliquot parts than ten, or any other number that is not much greater than itself. Twelve is divisible by *2*, *3*, *4* and *6*; and this circumstance fits it so well for the purpose of arithmetical computation that it has been resorted to in all times as the most convenient number into which any unit either of weight or measure could be divided. The divisions of the Roman *as*, the *libra*, the *jugerum*, and the modern foot, are all proofs of what is here asserted; and this advantage, which was perceived in rude and early times, would have been found of great value in the most improved eras of mathematical science. . . . We regret therefore that the experiment of this new arithmetic was not attempted. Another opportunity of trying it is not likely to occur soon.

"In the ordinary course of human affairs such improvements are not thought of, and the moment may never again present itself when the wisdom of a nation shall come up to the level of this species of reform."

If man had been created with six fingers on each hand, we would have had in arithmetic a duodenal instead of the present cumbrous decimal system.

A uniform duodenal system of metrology, even with decimal arithmetic, would be much better in the shop and market than the French metrical system.

A duodenal system would be equally applicable in all branches of metrology, and it would include those which are excluded by the metrical system—namely, astronomy, geography, navigation, time and the circle.

The duodenal system would require two new characters to represent *10* and *11*, so as to place *10* at *12*. This change in the figures would appear strange at the first glance, but a little reflection, with due consideration, would soon lead to the satisfaction that these two new figures simplify the arithmetic and render it much easier for mental calculation than decimal arithmetic.

Senidenal System.

The senidenal system has 16 to the base. A full elucidation of this system has been worked out by the author and was published in the year 1862 by J. B. Lippincott & Co., Philadelphia. It is called the tonal system.

The advantage of 16 as a base for arithmetic is that of its binary division to infinity. It is really the best system that could be devised for metrology and mental calculations.

The disadvantage of 16 as a base is that it requires six new figures to complete the base, which would be difficult to introduce, and also that it does not admit of trinary divisions, as is required in the circle and time, but it is under all circumstances far superior to the decimal system.

The difficulties with the decimal system are fully explained in the elucidation of the tonal system.

Scale of Four Arithmetical Systems.

Systems.	Base.	100	1000	10,000	100,000	1,000,000
Octonary.........	8	64	512	4,096	32,768	260,744
Denary...........	*10*	*100*	*1,000*	*10,000*	*100,000*	*1,000,000*
Duodenary......	12	144	1,728	20,736	238,832	2,865,984
Senidenary......	16	256	4,096	65,536	16,777,216	268,435,456

The names of the systems are Hindoo.

The octonary system requires the greatest number of places for expressing high numbers, for instance 1,000,000 octonal means only *260,744* of the decimal system.

The senidenary or tonal system uses less places; for instance, 1,000,000 senidenal means *268,435,456* of decimal numbers.

DUODENAL ARITHMETIC AND METROLOGY.

The base in the duodenal system is *12*, instead of *10* in the decimal system.

The Arabic system of notation is composed of ten simple digits, or characters—namely, *0*, *1*, *2*, *3*, *4*, *5*, *6*, *7*, *8*, *9*, and the base *10*.

These same characters can be used in the duodenal system by adding two numbers to complete the base—namely, *11* and *12*; then all the units of weights and measures should be divided and multiplied by *12*, but in order to render the system simple for calculation, it will be necessary to substitute new characters for the numbers *10*, *11* and *12*—namely,

Decimal system,	*1*, *2*, *3*, *4*, *5*, *6*, *7*, *8*, *9*, *10*, *11*, *12*;
Duodenary system,	1, 2, 3, 4, 5, 6, 7, 8, 9, ♀, ɤ, 10,

in which 10 denotes the base *12*, ♀ stands for *10*, and ɤ stands for *11*.

The Italic figures mean decimal numbers, and the Roman figures mean duodenal numbers.

In order to distinguish the two systems from one another, it will be necessary to give new names to the duodenary figures.

A duodenary system of arithmetic cannot be adopted by only one nation, but the whole civilized world ought to agree upon such a scheme. Different nations have different languages and names for the decimal figures and numbers; but in the adoption of a duodenary system of arithmetic, one common nomenclature might be agreed upon.

The new figures and nomenclature appear to be the greatest objection to the introduction of the duodenal system of arithmetic and metrology.

There is no difficulty in convincing the public of the utility of the duodenal system, and with that impression, a pride will be taken in using the new nomenclature, which could be taught in every school; and each individual would attempt to follow up the time of education.

The following table contains the names of the figures and numbers up to twelve in different languages:

Nomenclature of Numbers in Different Languages.

No.	English.	French.	German.	Swedish.	Spanish.	Latin.	Greek.
0	Naught.	Zéro.	Null.	Noll.	Cero.	Nihil.	Zero.
1	One.	Un.	Eins.	En.	Uno.	Unus.	Eis, En.
2	Two.	Deux.	Zwei.	Två.	Dos.	Duo.	Duo.
3	Three.	Trois.	Drei.	Tre.	Tres.	Tres.	Treis.
4	Four.	Quatre.	Vier.	Fyra.	Cuatro.	Quatuor.	Tessares.
5	Five.	Cinq.	Funf.	Fem.	Cinco.	Quinque.	Pente.
6	Six.	Six.	Sechs.	Sex.	Seis.	Sex.	Hex.
7	Seven.	Sept.	Siben.	Sju.	Siete.	Septem.	Hepta.
8	Eight.	Huit.	Acht.	Åtta.	Ocho.	Octo.	Okto.
9	Nine.	Neuf.	Neun.	Nio.	Nueve.	Novem.	Ennea.
10	Ten.	Dix.	Zehn.	Tio.	Diez.	Decem.	Deka.
11	Eleven.	Onze.	Elf.	Elva.	Once.	Undecem.	Endeka.
12	Twelve.	Douze.	Zwölf.	Tolf.	Doce.	Duodecem.	Dodeka.

No.	Russian.	Finnish.	Welsh.	Gaelic.	Gothic.	Turkish.	Hebrew.
0	Nill.	Nolli.				Zuffer.	Ayin.
1	Odna.	Yksi.	Una.	Unah.	Ains.	Bier.	Aleph.
2	Dva.	Kaksi.	Dou.	Gha.	Tvai.	Icki.	Beth.
3	Tri.	Koleme.	Tri.	Trec.	Threis.	Utch.	Gimel.
4	Tcheteri.	Nelja.	Pedvar.	Cheir.	Fidvor.	Duert.	Daleth.
5	Piat.	Viisi.	Pump.	Coag.	Finf.	Bach.	He.
6	Shest.	Kunsi.	Chewech.	Seach.	Saihs.	Altoe.	Vau.
7	Sem.	Seitsemän.	Saith.	Sheach.	Sibum.	Yedi.	Zain.
8	Vosem.	Kahdeksän.	Wyth.	Oacht.	Ahtan.	Seckiz.	Bheth.
9	Deviat.	Yhdeksän.	Nan.	Nuegh.	Niun.	Dokus.	Teth.
10	Desiat.	Kymmenän.	Deg.	Doach.	Taihun.	On.	Yod.
11	Odenatset.	Yksitoista.	Undeg.	Undech.	Ainstaihun.	Onbier.	Yodaleph.
12	Dvenatset.	Kaksitoista.	Doudeg.	Ghadech.	Tvaitaihun.	Onicki.	Yodbeth.

No.	Arabian.	Persian.	Hindoo.	Chinese.	Japanese.	Sanscrit.	Duodenal
0	Siforon.			Bow.	Ley.		Zero, 0
1	Ahed.	Yika.	Ek Ache.	Yat.	Itchi.	Aika.	An, 1
2	Ishnan.	Du.	Do.	Ge.	Ni.	Dwan.	Do, 2
3	Saylaset.	Seh.	Ien.	Sam.	San.	Tri.	Tre, 3
4	Erbayet.	Chehaur.	Châr.	Tze.	Tchi.	Chatur.	For, 4
5	Jemset.	Pendj.	Pânoh.	Ngnu.	Go.	Pancha.	Pat, 5
6	Sittet.	Shesh.	Chha.	Luck.	Lock.	Shash.	Sex, 6
7	Saybet.	Helft.	Sâth.	Tchut.	Sytchi.	Saptan.	Ben, 7
8	Saymaniet.	Hesht.	Ath.	Pbat.	Hatchi.	Ashta.	Ott, 8
9	Tiset.	Nuh.	Nau.	Geo.	Ku.	Navan.	Nev, 9
10	Eshret.	Deh.	Das.	Shop.	Dgiu.	Dashan.	Dis, ♈
11	Ahedeshere.	Yikadeh.	Gyârah.	Shopyat.	Dgiuitchi.	Aikadashan.	Elv, ♉
12	Ishnaneshere.	Dudeh.	Bârah.	Shopgaa.	Dgiuni.	Dwandashan.	Ton, 10

COMMENTS ON NOMENCLATURE.

On account of the different pronunciations of letters and words in different languages, the true sound of a name cannot be conceived without a knowledge of the language in which it is written.

The Japanese sound for 9 is written *ku* in the table, but for the English pronunciation it should be written *koo*.

There are some letters of the alphabet which have nearly the same sound in all languages, and only such letters should be used in the coining of names for the figures and numbers in the duodenary system.

The letters *th*, *w*, *o*, *ur*, *ght* in the English language, and also the letter *C*, which has two sounds in almost all languages, should not be used for the new names.

The names given to the duodenary figures in the last column are clear and distinct sounds, which would be well understood and pronounced alike in all languages.

It would be useless to attempt to introduce the names of the figures and numbers in either of the languages above given as a universal nomenclature, for not only that they are not suited for more than the language in which they are written, but prejudices would be against them. The introduction of the French metrical system has been greatly retarded by reason merely of its cumbersome nomenclature.

The best work on the etymology of numbers known to the author is that of Professor S. Zehetmayr, published in Leipsic, 1854. In the establishment of a new and universal nomenclature of numbers we ought to select clear and distinct sounds, which can be understood and pronounced alike in all languages, without regard to the etymology of numbers.

The Arabic notation of numbers is yet used only by about one-third of the population of the earth, and the other two-thirds use different kinds of irregular characters or hieroglyphics, which combinations are unfit for arithmetical calculations.

The Roman notation was used in England up to the beginning of the seventeenth century, when the Arabic notation was gradually gaining ground against very strong opposition; and at last caused the burning of the houses of Parliament. The Arabic notation was introduced into Germany in the twelfth century, and into Italy in the eleventh century.

Comparison of Numbers in the Duodenary and Decimal Systems, with the Corresponding New Names.

New.	Names.	Old.	New.	Names.	Old.
0	Zero	*0*	37	Tretoben	*43*
1	An	*1*	38	Tretonott	*44*
2	Do	*2*	39	Tretonev	*45*
3	Tre	*3*	3♀	Tretondis	*46*
4	For	*4*	3♉	Tretonelv	*47*
5	Pat	*5*	40	Forton	*48*
6	Sex	*6*	41	Fortonan	*49*
7	Ben	*7*	42	Fortondo	*50*
8	Ott	*8*	43	Fortontre	*51*
9	Nev	*9*	44	Fortonfor	*52*
♀	Dis	*10*	45	Fortonpat	*53*
♉	Elv	*11*	46	Fortonsex	*54*
10	Ton	*12*	47	Fortoben	*55*
11	Tonan	*13*	48	Fortonott	*56*
12	Tondo	*14*	49	Fortonev	*57*
13	Tontre	*15*	4♀	Fortondis	*58*
14	Tonfor	*16*	4♉	Fortonelv	*59*
15	Tonpat	*17*	50	Paton	*60*
16	Tonsex	*18*	51	Patonan	*61*
17	Toben	*19*	52	Patondo	*62*
18	Tonott	*20*	53	Patontre	*63*
19	Tonev	*21*	54	Patonfor	*64*
1♀	Tondis	*22*	55	Patonpat	*65*
1♉	Tonelv	*23*	56	Patonsex	*66*
20	Doton	*24*	57	Patoben	*67*
21	Dotonan	*25*	58	Patonott	*68*
22	Dotondo	*26*	59	Patonev	*69*
23	Dotontre	*27*	5♀	Patondis	*70*
24	Dotonfor	*28*	5♉	Patonelv	*71*
25	Dotonpat	*29*	60	Sexton	*72*
26	Dotonsex	*30*	61	Sextonan	*73*
27	Dotoben	*31*	62	Sextondo	*74*
28	Detonott	*32*	63	Sextontre	*75*
29	Detonev	*33*	64	Sextonfor	*76*
2♀	Dotondis	*34*	65	Sextonpat	*77*
2♉	Dotonelv	*35*	66	Sextonsex	*78*
30	Treton	*36*	67	Sextoben	*79*
31	Tretonan	*37*	68	Sextonott	*80*
32	Tretondo	*38*	69	Sextonev	*81*
33	Tretontre	*39*	6♀	Sextondis	*82*
34	Tretonfor	*40*	6♉	Sextonelv	*83*
35	Tretonpat	*41*	70	Benton	*84*
36	Tretonsex	*42*	71	Bentonan	*85*

New.	Names.	Old.	New.	Names.	Old.
72	Bentondo..........	*86*	♉1	Elvtonan..........	*133*
73	Bentontre	*87*	♉2	Elvtondo..........	*134*
74	Bentonfor.........	*88*	♉3	Elvtontre	*135*
75	Bentonpat	*89*	♉4	Elvtonfor.........	*136*
76	Bentonsex.........	*90*	♉5	Elvtonpat.........	*137*
77	Bentoben..........	*91*	♉6	Elvtonsex.........	*138*
78	Bentonott.........	*92*	♉7	Elvtoben	*139*
79	Bentonev..........	*93*	♉8	Elvtonott.........	*140*
7♈	Bentondis.........	*94*	♉9	Elvtonev	*141*
7♉	Bentonelv	*95*	♉♈	Elvtondis.........	*142*
80	Otton	*96*	♉♉	Elvtonelv.........	*143*
81	Ottonan	*97*	100	San..................	*144*
82	Ottondo	*98*	148	San-fortonott.....	*200*
83	Ottontre...........	*99*	200	Dosan..............	*288*
84	Ottonfor...........	*100*	210	Dosan-ton	*300*
85	Ottonpat...........	*101*	300	Tresan.............	*432*
86	Ottonsex...........	*102*	358	Tresan-patonott..	*500*
87	Ottoben............	*103*	400	Forsan.............	*576*
88	Ottonott	*104*	420	Forsan-doton.....	*600*
89	Ottonev	*105*	500	Patsan.............	*720*
8♈	Ottondis...........	*106*	568	Patsan-sextonott.	*800*
8♉	Ottonelv...	*107*	600	Sexan..............	*864*
90	Nevton............	*108*	630	Sexan-treton......	*900*
91	Nevtonan.........	*109*	700	Bensan............	*1008*
92	Nevtondo.........	*110*	800	Ottsan	*1152*
93	Nevtontre.........	*111*	900	Nevsan............	*1296*
94	Nevtonfor.........	*112*	♈00	Dissan	*1440*
95	Nevtonpat.........	*113*	♉00	Elvsan.............	*1584*
96	Nevtonsex.........	*114*	1000	Tos..................	*1728*
97	Nevtoben.........	*115*	1100	Tossan	*1872*
98	Nevtonott.........	*116*	1200	Tosdosan..........	*2016*
99	Nevtonev..........	*117*	1300	Tostresan	*2160*
9♈	Nevtondis.........	*118*	1400	Tosforsan..........	*2304*
9♉	Nevtonelv.........	*119*	1500	Tospatsan.........	*2448*
♈0	Diston..............	*120*	1600	Tossexan	*2592*
♈1	Distonan...........	*121*	1700	Tosbensan.........	*2736*
♈2	Distondo...........	*122*	1800	Tosottsan..........	*2880*
♈3	Distontre..........	*123*	1900	Tosnevsan.........	*3024*
♈4	Distonfor	*124*	1♈00	Tosdissan	*3168*
♈5	Distonpat.........	*125*	1♉00	Toselvsan.........	*3312*
♈6	Distonsex.........	*126*	2000	Dotos..............	*3456*
♈7	Distoben...........	*127*	4000	Fortos..............	*6912*
♈8	Distonott	*128*	6000	Sextos..............	*10368*
♈9	Distonev...........	*129*	8000	Ottos................	*13724*
♈♈	Distondis..........	*130*	♈000	Distos..............	*17180*
♈♉	Distonelv	*131*	10000	Dill	*20736*
♉0	Elvton.............	*132*			

FRACTIONS.

Duodenary System.	Decimal System.
1/2 = 0.6	1/2 = *0.5*
1/4 = 0.3	1/4 = *0.25*
3/4 = 0.9	3/4 = *0.75*
1/8 = 0.16	1/8 = *0.125*
3/8 = 0.46	3/8 = *0.375*
5/8 = 0.76	5/8 = *0.625*
7/8 = 0.♀6	7/8 = *0.875*
1/3 = 0.4	1/3 = *0.3333. . . . . . .*
2/3 = 0.8	2/3 = *0.6666. . . . . . .*
1/6 = 0.2	1/6 = *0.16666. . . . . .*
5/6 = 0.♀	5/6 = *0.83333. . . . . .*
1/14 = 0.09	1/16 = *0.0625*
3/14 = 0.23	3/8 = *0.1875*
7/14 = 0.53	7/16 = *0.4375*
Ծ/14 = 0.83	11/16 = *0.6875*
1/20 = 0.06	1/24 = *0.041666. . . . . .*
7/20 = 0.36	7/24 = *0.2916666. . . . .*
Ծ/20 = 0.56	11/24 = *0.458333. . . . . .*
1/28 = 0.046	1/32 = *0.03125*
7/28 = 0.976	7/32 = *0.21875*
1/9 = 0.14	1/9 = *0.11111111. . . .*
2/9 = 0.24	2/9 = *0.22222. . . . . .*
4/9 = 0.54	4/9 = *0.44444. . . . . .*
5/9 = 0.68	5/9 = *0.55555. . . . . .*
8/9 = 0.♀8	8/9 = *0.88888. . . . . .*
Ծ/28 = 0.416	11/32 = *0.34375*
17/28 = 0.646	17/32 = *0.53125*
1/54 = 0.023	1/64 = *0.015625*
Ծ/54 = 0.209	11/64 = *0.171875*
33/54 = 0.739	39/64 = *0.609375*

The above table of fractions shows the simplicity of the duodenary system, which requires few figures where the old system requires a

great number of decimals. For 3ds, 6ths, 9ths, 12ths and 24ths the duodenary system finishes the fraction with one or two places where the number of decimals is endless.

Addition Table.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**♀**	**♉**	**10**
2	4	5	6	7	8	9	♀	♉	10	11	12
3	5	6	7	8	9	♀	♉	10	11	12	13
4	6	7	8	9	♀	♉	10	11	12	13	14
5	7	8	9	♀	♉	10	11	12	13	14	15
6	8	9	♀	♉	10	11	12	13	14	15	16
7	9	♀	♉	10	11	12	13	14	15	16	17
8	♀	♉	10	11	12	13	14	15	16	17	18
9	♉	10	11	12	13	14	15	16	17	18	19
♀	10	11	12	13	14	15	16	17	18	19	1♀
♉	11	12	13	14	15	16	17	18	19	1♀	1♉
10	12	13	14	15	16	17	18	19	1♀	1♉	20

Multiplication Table.

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**♀**	**♉**	**10**
2	4	[illegible]	8	♀	10	12	14	16	18	1♀	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	♀	13	18	21	26	2♉	34	39	42	47	50
6	10	16	20	26	20	36	40	46	50	56	60
7	12	19	24	2♉	36	41	48	53	5♀	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
♀	18	26	34	42	50	5♀	68	76	84	92	♀0
♉	1♀	29	38	47	56	65	74	83	92	♀1	♉0
10	20	30	40	50	60	70	80	90	♀0	♉0	100

The duodenal multiplication table of the single figures is 44 per cent. more extensive than that of the decimal, but the binary and trinary properties makes it much easier to learn and to remember.

Examples in Addition.

38♀45℧3	8♀093	0.03
6108℧♀5	9℧6♀	℧3.06
99℧1598	♀3℧	4♀9.♀1
	3♀	674℧.60
	98♀℧♀	39♀06.00
		44℧36.4♀

Examples in Subtraction.

74℧8♀6	3♀43.℧1	0.℧48♀6
314364	1℧9♀.0♀	0.003℧♀
437542	1♀65.♀3	0.℧44♀8

Examples in Multiplication.

8694	♀4℧63	36℧.3♀45
24	63℧	0.0℧6
2♀314	956689	1957℧226
5168	272♀69	32442507
7℧994	525916	34.19♀4296
	55♀36759	

Examples in Division.

42)136℧0♀(38♀4	4♀.℧)3♀057.63♀(946.38
106	3823
30℧	1♀27
294	1778
370	26℧6
358	2556
14♀	1603
148	1289
002	3364
	333♀
	36

On account of the binary and trinary properties of the duodenary system, these arithmetical operations are much easier to the mind than those with decimal arithmetic. The only difficulty about it is to unlearn the decimal system.

The duodenary system has all the advantages and none of the disadvantages of the decimal system; it is also better adapted to mental calculations, which are very difficult with our present arithmetic.

METROLOGY.

The utility of a duodenary system of arithmetic consists in its combination with a similar system of metrology—namely, that all units of measure should be divided and multiplied by the same base, twelve.

Units of measure are required for the following fifteen quantities.

Length.	Weight.	Heat.	Force.	Power.
Surface.	Mass.	Light.	Velocity.	Space.
Volume.	Money.	Electricity.	Time.	Work.

Measurement of Length.

Assume the mean circumference of the earth to be the primary unit of length, and divide it by *twelve* repeatedly until the divisions are reduced to a length which would be a convenient unit to handle in the shop and in the market.

The mean circumference of the earth is about *24851.64* miles, which, multiplied by *5280*, will be

		Duodenal.
0	*131216659.2* feet....	1 circum.
1	*10934721.6* feet..............	1 hour.
2	*911226.8* feet..............	1 grad.
3	*75935.56* feet............	1 minute.
4	*6327.96* feet............	1 mile.
5	*527.33* feet............	1 cable.
6	*43.944* feet...........	1 chain.
7	*3.772* feet...........	1 metre.
The required unit of length	*43.944* inches........	1 metre.

The length of the circumference of the earth, divided by the seventh power of *12*, gives a length of *43.944* inches, which is assumed as a unit for all measurements of length, and which we will call a **metre.**

Twelve duodenal metres is a length of *43.944* feet, which is a convenient measure in the field or in surveys, and which we will call a **chain.**

Twelve duodenal chains is a length of *527.33* feet, which we will call one **cable.**

Twelve duodenal cables is a length of *6327.96* feet, which we will call one **mile.** The duodenal mile will be about *300* feet longer than our present knot or sea-mile.

Twelve duodenal miles = 1 **minute,**
Twelve duodenal minutes = 1 **grad,**
Twelve duodenal grads = 1 **hour,**
Twelve duodenal hours = 1 **circum,**
on the earth's great circle.

The duodenal metre to be divided into twelve equal parts of *3.772* inches each, and called **metons**. The meton into twelve equal parts of *0.31433* of an inch each, called **mesans**. The mesan into twelve equal parts of *0.0262* of an inch each, called **metos**.

Fig. 235 shows the full size of a meton with its divisions.

Fig. 235.

The first 6 **mesans** are divided into **metos**, and the last into quarters of **mesans**. The ordinary shop-metre need not be divided finer than into quarters of **mesans**, for in so small divisions the **metos** can easily be approximated.

The **metons** and **mesans** would be the most convenient for expressing short measures in the mechanic arts.

Fig. 236. Fig. 240.

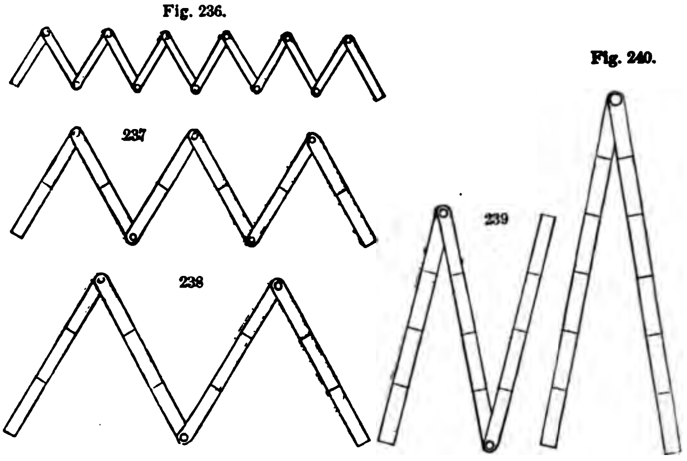


Fig. 236 represents a twelve-folded duodenal metre with lap-joints, like the ten-folded French metre; each part is one meton of *3.772* inches.

Fig. 237 represents a six-folded duodenal metre with lap-joints, of *7.544* inches in each. This form could be made with regular hinges like the English rule.

Fig. 238 represents a four-folded duodenal metre, with 3 metons in each part of *11.316* inches. This would be the most convenient form for the shop when folded with regular hinges like the English four-folded rule.

Fig. 239 represents a three-folded metre, with four metons in each part of *15.088* inches.

Fig. 240 represents a two-folded metre, with six metons in each part of about *22* inches.

We see here that the duodenal metre can be folded into five different forms, with even measures in each part.

The longest unit of measure is the circumference of the earth, which ought to be termed a **circum.** The circum should be used in expressing astronomical distances.

The duodenal grad is 100 duodenal miles, or 0.01 of the earth's great circle, which would be a proper measure for expressing long distances on the earth's surface; and which would convey a correct idea of the real magnitude of such distances compared with the great circle.

The mile would be the common road measure and for traveling distances on land and sea.

Duodenal Measures of Length.

Circum.	Grad.	Mile.	Cable.	Chain.	Metre.	Meton.	Mesan.	Metos.
1	100	10000	100000	1000000				
0.01	1	100	1000	10000	100000	1000000		
0.0001	0.01	1	10	1000	10000	100000	1000000	
	0.0001	0.1	1	10	1000	10000	100000	1000000
	0.000001	0.01	0.1	1	10	1000	10000	100000
		0.001	0.01	0.1	1	10	1000	10000
		0.0001	0.001	0.01	0.1	1	10	1000
		0.00001	0.0001	0.001	0.01	0.1	1	10
			0.00001	0.0001	0.001	0.01	0.1	1

Division of the Circle.

The circle to be divided into 100 equal parts (144 decimal).

Duodenal System.		*Old System.*
One circle	= 100 grads.	*360* degrees.
One grad	= 100 lents.	*2* degrees *30* minutes.
One lent	= 100 ponts.	*1* minute *2.5* seconds.
One pont	=	*0.43418* of a second.
One quadrant	= 30 grads.	*90* degrees.

One duodenal mile on the earth's surface corresponds with an angle of one lent.

One duodenal chain on the earth's surface corresponds with an angle of one pont.

The latitude and longitude to be divided as the circle.

The angular measures correspond with the linear measures on the earth's surface. The terms minute and second are omitted in the division of the circle, so as not to confound angles with time.

The circle can thus be divided into *2*, *3*, *4*, *6*, *8*, *9*, *12* or *16* parts, without leaving fractions of a degree or grad.

The quadrant of the circle, containing 30 grads (*36*), can be divided into *2*, *3*, *4*, *6*, *9* or *12* parts without leaving fractions of a grad. These advantages with the duodenal division of the circle are of great importance in geometry, geography, trigonometry, astronomy and in navigation.

Either of the divisions corresponds with an even linear measure on the earth's surface.

Duodenal Division of Time.

The division of time should conform to that of the circle.

The time from noon to noon, including one night and day, to be divided into twelve equal parts, called hours.

	Duodenal System.		*Old System.*
1.	One day	= 10 hours.	*24* hours.
	One hour	= 10 grads.	*2* hours.
	One grad	= 10 minutes.	*10* minutes.
	One minute	= 10 lents.	*0.83333* of a minute.
	One lent	= 10 seconds.	*4.1666* seconds.
	One second	= 10 ponts.	*0.3472* of a second.

2.	One day	= 10 hours.
	One hour	= 100 minutes.
	One minute	= 100 seconds.

3.	One day	= 100 grads.
	One grad	= 100 lents.
	One lent	= 100 ponts.

Either of these three divisions can be used in practice. The first division includes the second and third.

If the duodenal division of time was introduced all over the world, some nations would probably use the second expression, and others the third, but the third division is the best, because the hands on the watch would show the number of grads.

In the notation of time, say 3 hours and 46 minutes, will appear 3.46 hours, or 34.6 grads, or 346 minutes.

5 hours, 36 minutes and 15 seconds will appear 5.36.15 hours, or 53.61.5 grads.

The conversion of angle into time, or time into angle, is only to move the point one place.

There is no necessity of *A. M.* and *P. M.* in the duodenal time.

Astronomers would surely use the third expression of time, which corresponds with the divisions of the circle.

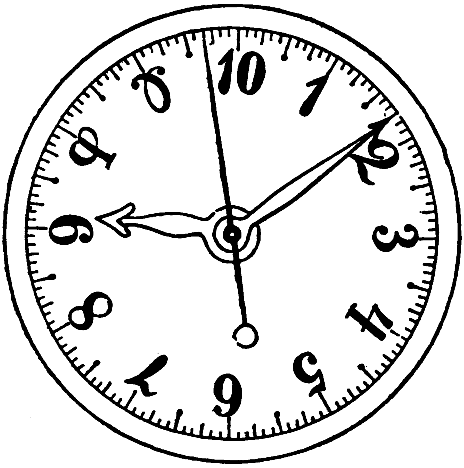
Duodenal Clock-dial.

Fig. 241 represents a duodenal clock-dial.

The hour-hand makes one turn in one night and day.

The minute-hand goes round once per hour, and the second-hand once per minute.

Fig. 241.



The hour-hand will point to 10 at noon, to 3 at *6* o'clock in the evening, to 6 at midnight, and to 9 at *6* o'clock in the morning.

The length of the pendulum vibrating duodenal seconds will be

$$l = 39.1 \times 0.3472^2 = 4.711 \text{ inches, or}$$
$$= 1.3 \text{ metons.}$$

The duodenal metre will vibrate

$$n = \frac{6.254 \times 60}{\sqrt{43.944}} = 56.6 \text{ times per old minute.}$$
$$= 41.55 \text{ times per duodenal minute.}$$

Duodenal Year.

The year is already divided into twelve months, but the division is unnecessarily irregular.

No.	Days.	Months.	Days.	Old.
1	26	January,	*30*	*31*
2	26 †	February,	*30**	*28**
3	26	March,	*30*	*31*
4	27	April,	*31*	*30*
5	26	May,	*30*	*31*
6	27	June,	*31*	*30*
7	27	July,	*31*	*31*
8	26	August,	*30*	*31*
9	27	September,	*31*	*30*
⚲	26	October,	*30*	*31*
Ɛ	27	November,	*31*	*30*
10	26	December,	*30*	*31*
	265	Year.	*365*	*365*

The days in the year ought to be divided so as to make the months of nearly equal lengths.

The two months following one another—namely, December and January—have both *31* days, and then comes February with only *28* days.

There is no good reason why the months should not be divided so as to have *30* days in seven months and 31 days in five months of the year, as shown by the accompanying table.

Different calendars are also used in different parts of the world, which ought to be only one common calendar.

* In leap years February should have *31* days, or † *27* duodenal.

Duodenal Compass.

The compass to be divided into grads like the circle, but numbered from North and South toward East and West, making 30 grads in each quadrant. Fig. 242 represents a duodenal compass.

The hours 1 and 2, corresponding each with 10 grads, are marked on the dial in each quadrant.

The nomenclature will be nearly the same as for the old compass, only the expression of fractional points would be changed to grads; for example, **South South-East, one-half South,** would be called simply **South ott East.**

Our present compass is divided into *32* points, and each point into four quarters, making *32* divisions in each quadrant, which shows the natural tendency toward binary divisions; but it is accompanied with a clumsy nomenclature. A course of 3¼ points from North toward East is termed **North-East by North, one-quarter East.** The duodenal expression would be simply **North an tre Est,** meaning one hour and three grads from North toward East, without expression of fractions; and the course is given with greater precision than by the present nomenclature.

Fig. 242.

Duodenal Measurement of Surface.

Small surfaces can be expressed in square metres, square metons or square mesans.

Duodenal System.		Old System.
One square chain	= 1 lot.	
10 chains square	= 1 acre.	
One square cable	= 1 acre.	6.3925 acres.
One acre	= 100 lots.	278075 square feet.
One lot	= 100 square metres.	1931.1 square feet.
One square mile	= 100 acres.	920.52 acres.

One square grad = 10,000 square miles.
One square grad = 1,000,000 acres.

Duodenal Measure of Capacity.

The cubic metre to be the unit for capacity.

Duodenal System.		Old System.
One cubic metre	= 1 tun.	49.113 cubic feet.
One tun	= 10 barrels.	49.113 cubic feet.
One barrel	= 10 pecks.	4.0927 cubic feet.
One peck	= 10 gallons.	643.92 cubic inches.
One gallon	= 10 glasses.	53.66 cubic inches.
One glass	= 10 spoons.	4.47 cubic inches.

The duodenal gallon is one cubic meton, or about one quart. An ordinary quart bottle would contain one duodenal gallon.

Dry and wet measures of capacity should be measured by the same units. A cord of wood 10 cubic metres.

The volume of solids should be measured by the cube of the linear units.

Duodenal Division of Money.

The unit of money ought to be the value of one duodenal dram of fine gold, which is about one dollar.

Duodenal System.	*American Money.*
One dollar = 10 shillings.	*1* dollar.
One shilling = 10 cents.	*8.3333* cents.
One cent =	*0.7* of a cent.

The American dollar is divided into ten dimes, but that expression is rarely used in the market. The same is the case with the French franc and dixiéme. The reason of that is that the decimal base does not admit of binary divisions. In a duodenal system the name of a twelfth part of a dollar would be used.

Dolls. Cts.	*Dolls. Cts.*	*Dolls. Cts.*
1/2 = 60	7/8 = φ6	3/14 = 23
1/4 = 30	1/3 = 40	7/14 = 53
3/4 = 90	2/3 = 80	Ɛ/14 = 83
1/8 = 16	1/6 = 20	1/20 = 6
3/8 = 46	5/6 = φ0	7/20 = 36
5/8 = 76	1/14 = 9	Ɛ/20 = 56

The 14ths in the duodenal system are the same as *16*ths in decimals.

The 20ths duodenal are 24ths decimal.

The duodenal system admits of binary division of the dollar as far as required in commerce and in the market.

Duodenal Measure of Weight.

The weight of one cubic metre of distilled water is assumed to be the unit of weight, and called one *ton*.

The duodenal ton will weigh about *3063.8* pounds, or *1.368* old tons.

Duodenal System.		*Old System Avoirdupois.*
One ton	= 10 pud.	*3063.8* pounds avoirdupois.
One pud	= 10 vegts.	*255.3166* " "
One vegt	= 10 ponds.	*21.276* " "
One pond	= 10 ounces.	*1.773* " "
One ounce	= 10 drachms.	*2.3640* ounces "
One drachm	= 10 scruples.	*0.1969* " "
One scruple	= 10 grains.	*0.0164* " "
One grain	=	*0.598* grains Troy.

Ton.	Pud.	Vegt.	Pond.	Ounce.	Dram.	Scruple.
1	10	100	1,000	10,000	100,000	1,000,000
0.1	1	10	100	1,000	10,000	100,000
0.01	0.1	1	10	100	1,000	10,000
0.001	0.01	0.1	1	10	100	1,000
0.0001	0.001	0.01	0.1	1	10	100
0.00001	0.0001	0.001	0.01	0.1	1	10
0.0000001	0.00001	0.0001	0.001	0.01	0.1	1

Units of Force.

Force can be measured by either one of the units of weight.

The pond would be the most convenient unit in estimating power and work in machinery.

Unit of Velocity.

Metons per second would be the most appropriate expression of velocity in machinery.

A velocity of metons per second is the same as miles per hour.

Unit of Time.

The second is the best unit of time to be used in the operation of machinery and falling bodies.

Unit of Power.

A force of one pond moving with a velocity of one meton per second to be one unit of power, and called **Effect.**

A power of one pond moving with a velocity of one meton per second would be $= 1.605$ foot-pounds per old second. This will make 30 duodenal effects per man-power, and 300 effects per horse-power.

Unit of Space.

The unit of linear space in the operation of machinery should be the meton or metre.

Unit of Work.

The work of lifting one ton through a height of one metre is a proper unit for estimating heavy work; it is equal to *11375* foot-pounds. This unit should be termed metreton and be used in the estimate of work of heavy ordnance.

The work which a laborer can accomplish per day would be about 100 metretons, which unit ought to be called a **Workmanday.**

The unit of work corresponding to velocity and effects should be one pond lifted one meton, which is *0.5567* of a foot-pound.

Unit of Mass.

The duodenal unit of mass would be the amount of matter in one cubic meton of distilled water, to be called one **Matt,** which is *53.668* cubic inches of water.

Unit of Gravity.

The velocity which a falling body would attain at the end of the first duodenal second is $g =$ 2.δ33 metres per second, which would be the acceleratrix of gravity.

Unit of Temperature.

The thermometer scale should be divided into 100 duodenal parts (*144*) between the freezing and boiling points of distilled water at the level of the sea in latitude 16 grads (45°).

One duodenal grad $= 1.25°$ Fahrenheit scale.

One duodenal grad $= 0.69°$ Centigrade.

Unit of Heat.

The heat required to raise the temperature of one pond of distilled water from 𝔓° to 𝔛° to be one unit of heat, which answers to *1713* foot-pounds of work.

Each kind of measure has different grades of units varying with the duodenal base, and any one of the units divided by 2, 3, 4 or 6 gives aliquot numbers in the quotient, which property renders the duodenal system very easy and clear to the mind for mental calculations and estimations of quantities.

In the establishment of a duodenal system of arithmetic and metrology it would perhaps be best to introduce the metrology first, and work it with decimal arithmetics until fairly established, after which the duodenal arithmetic would become more easy to learn and to apply.

The transition would not last long, for when one becomes imbued with the advantages and simplicity of the duodenal principles he would not bother his brain any more with the unnatural decimal base, but encourage others to take up the new system.

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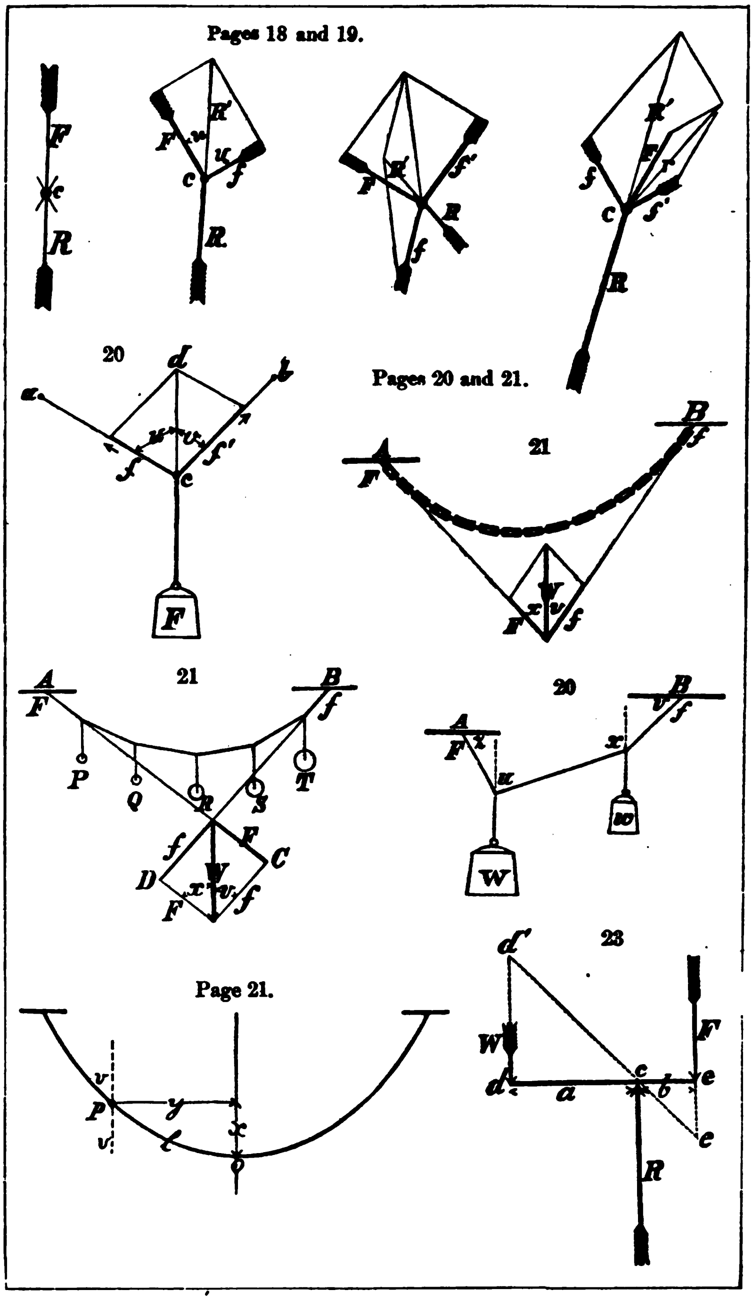
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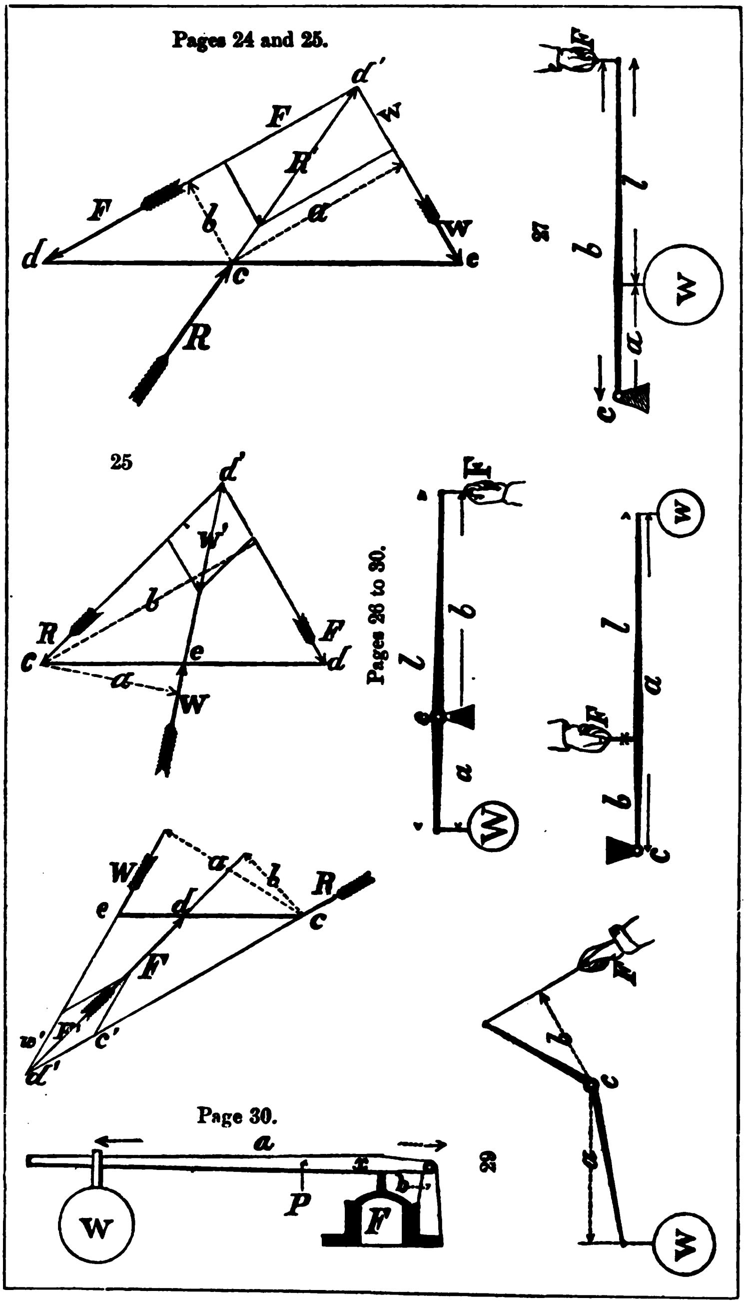
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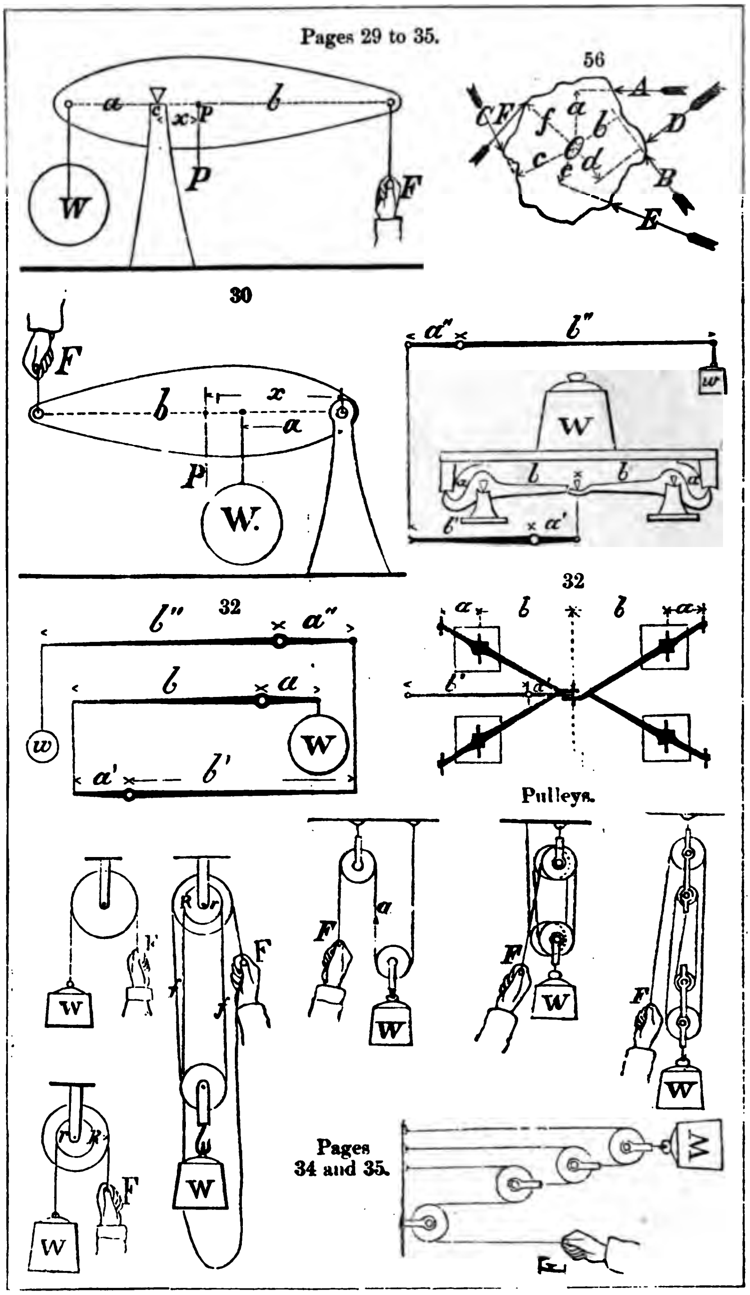
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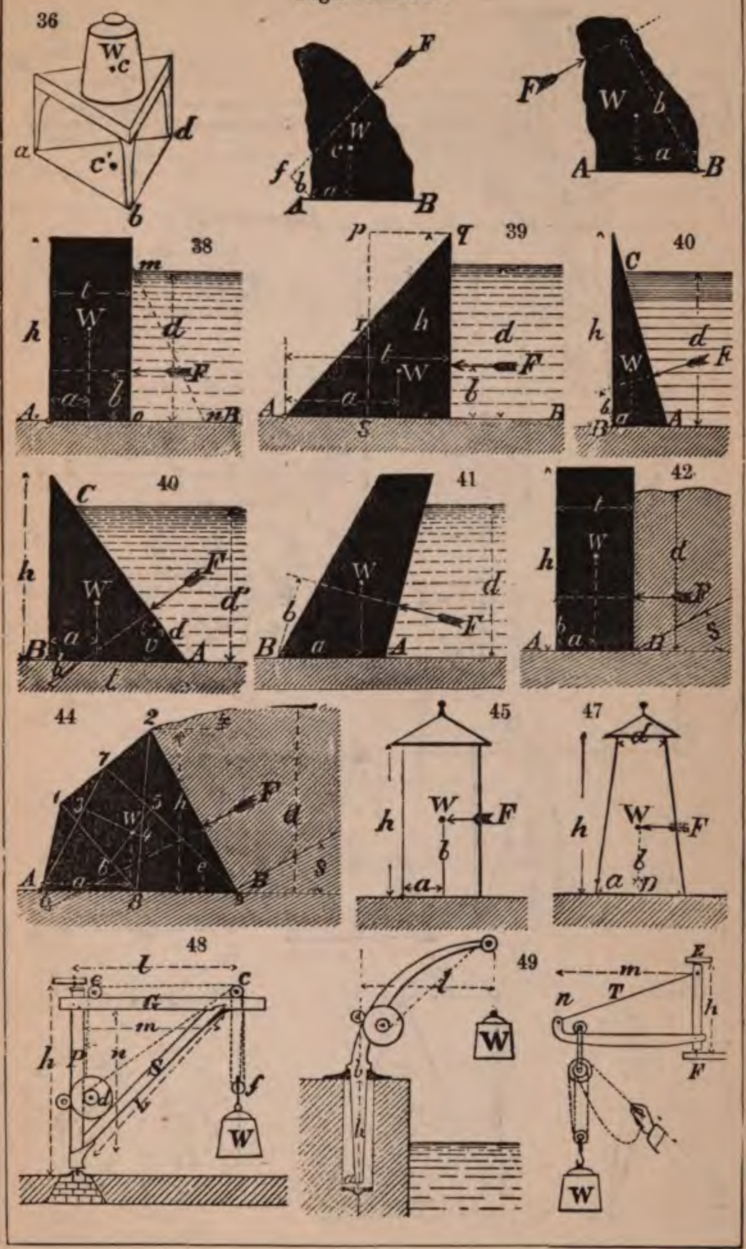

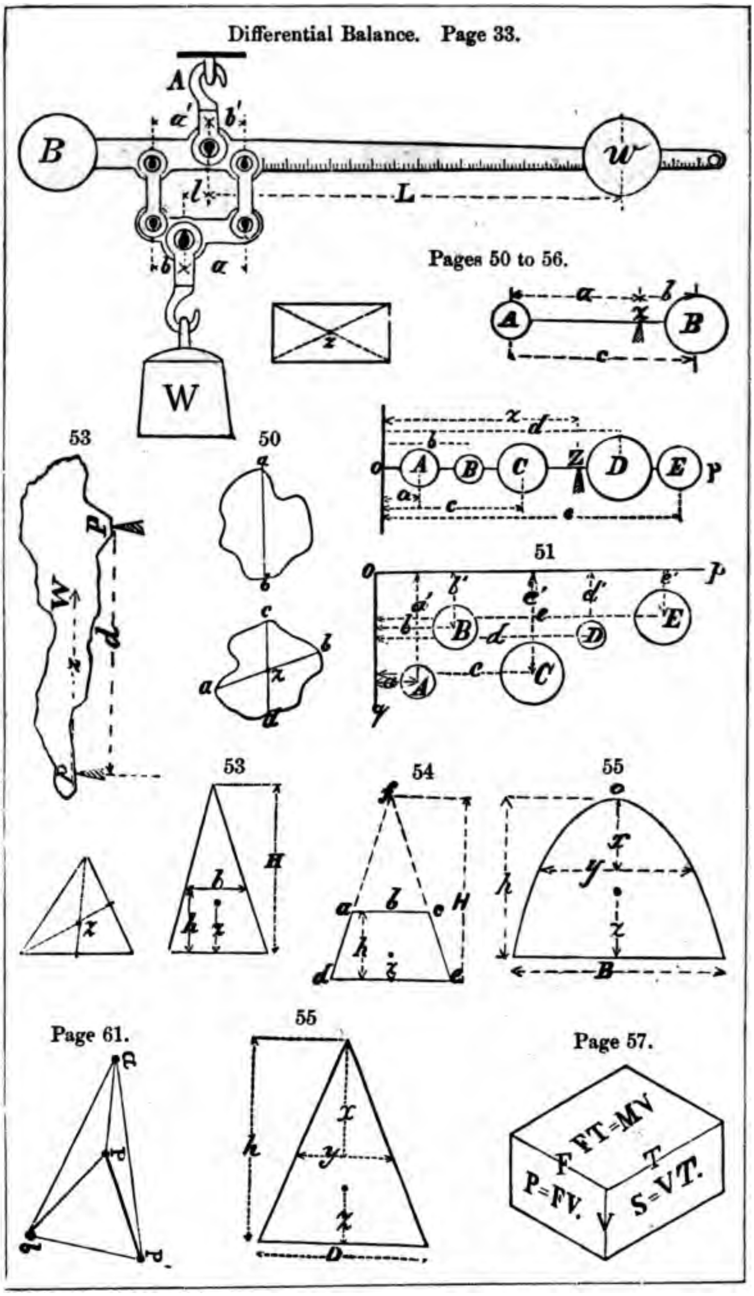
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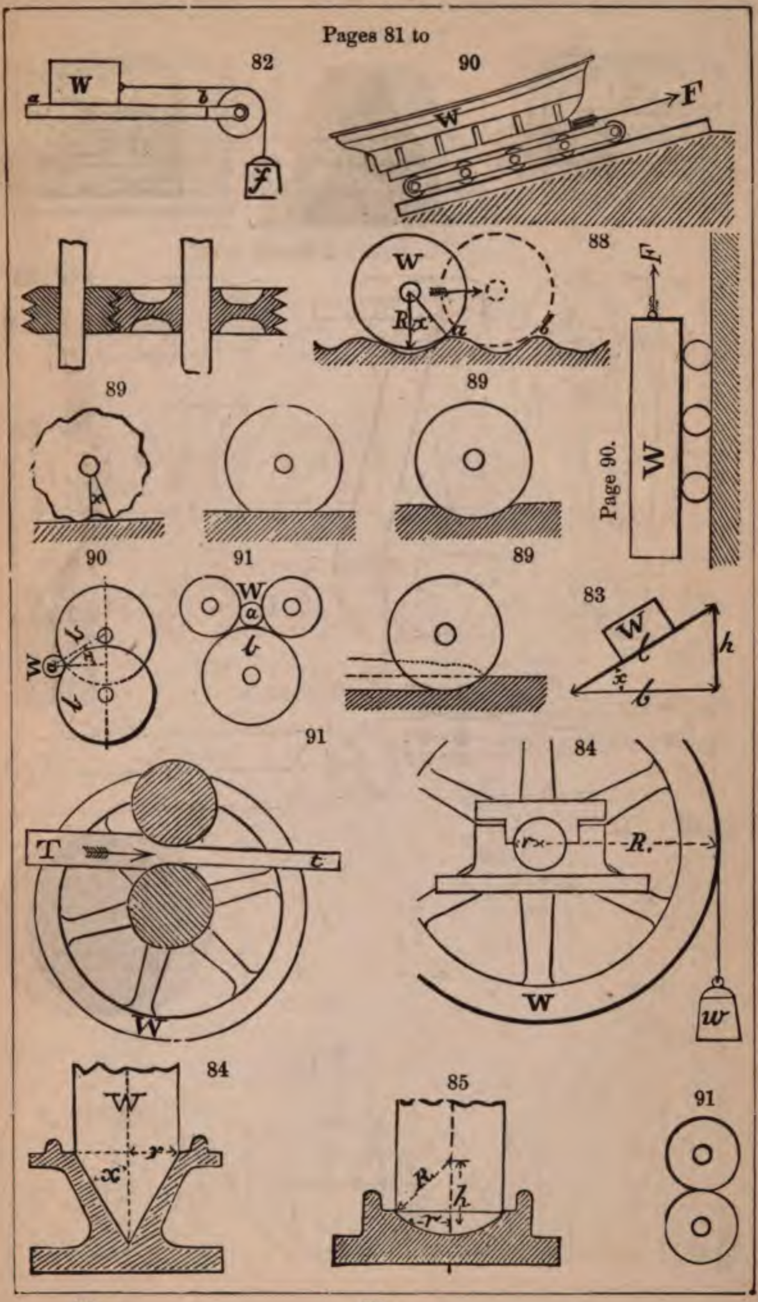
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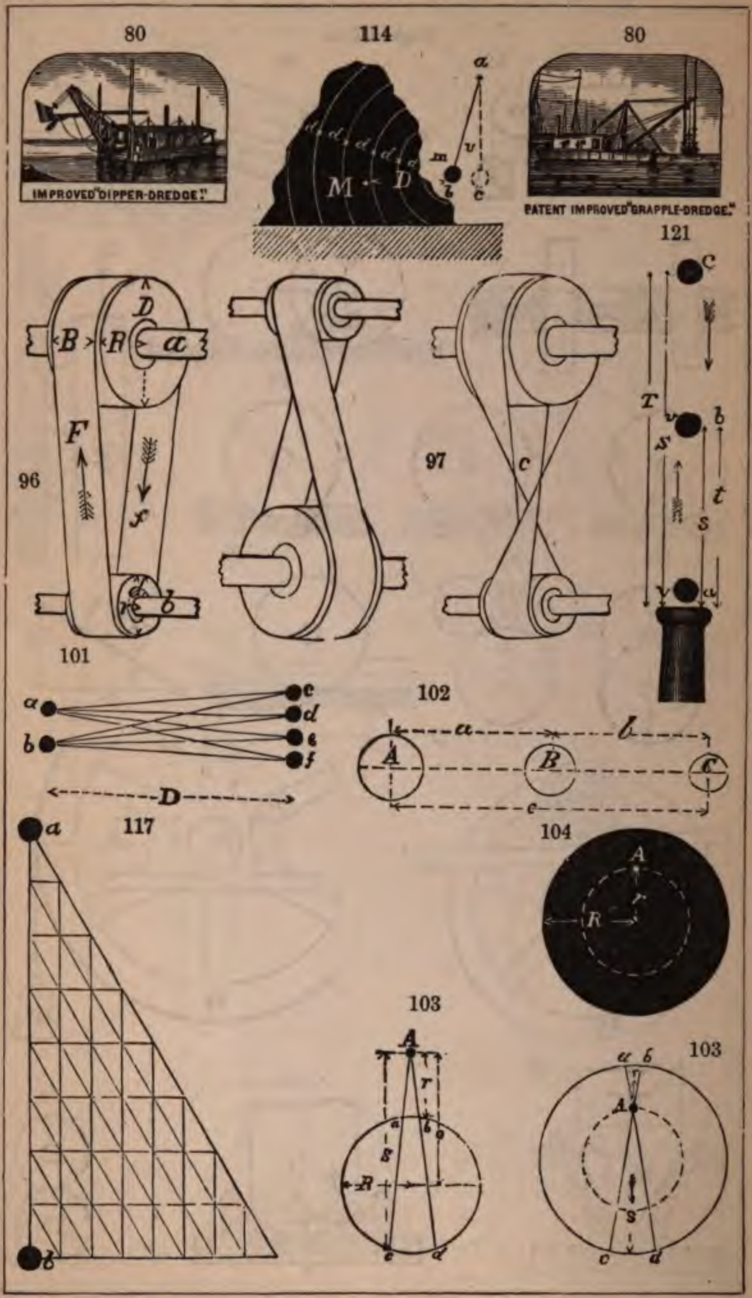
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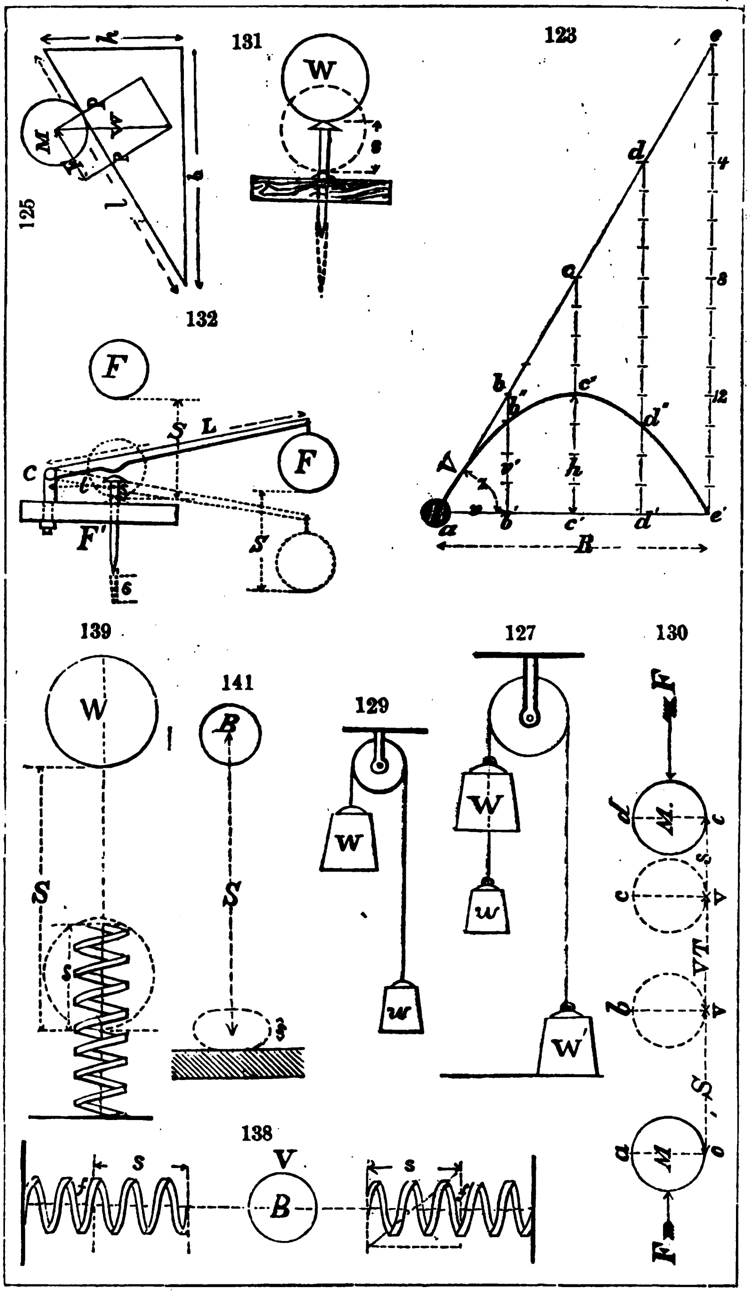
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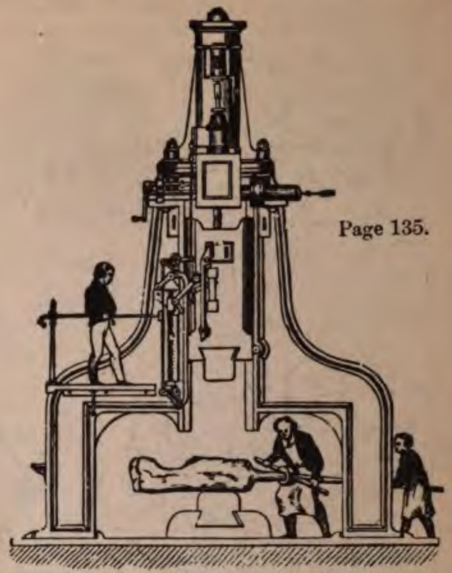

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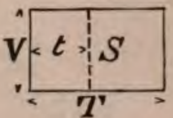

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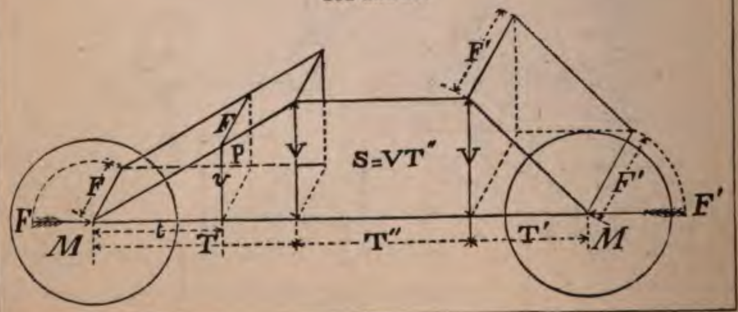

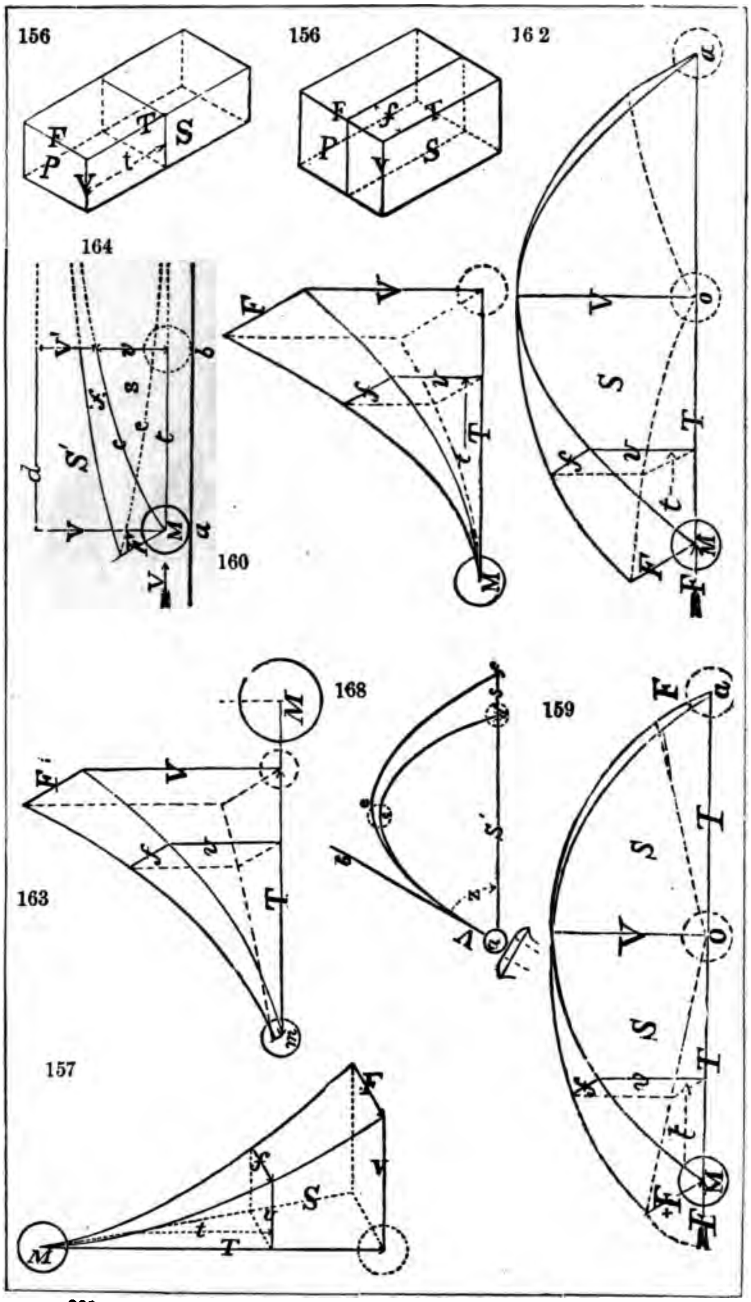
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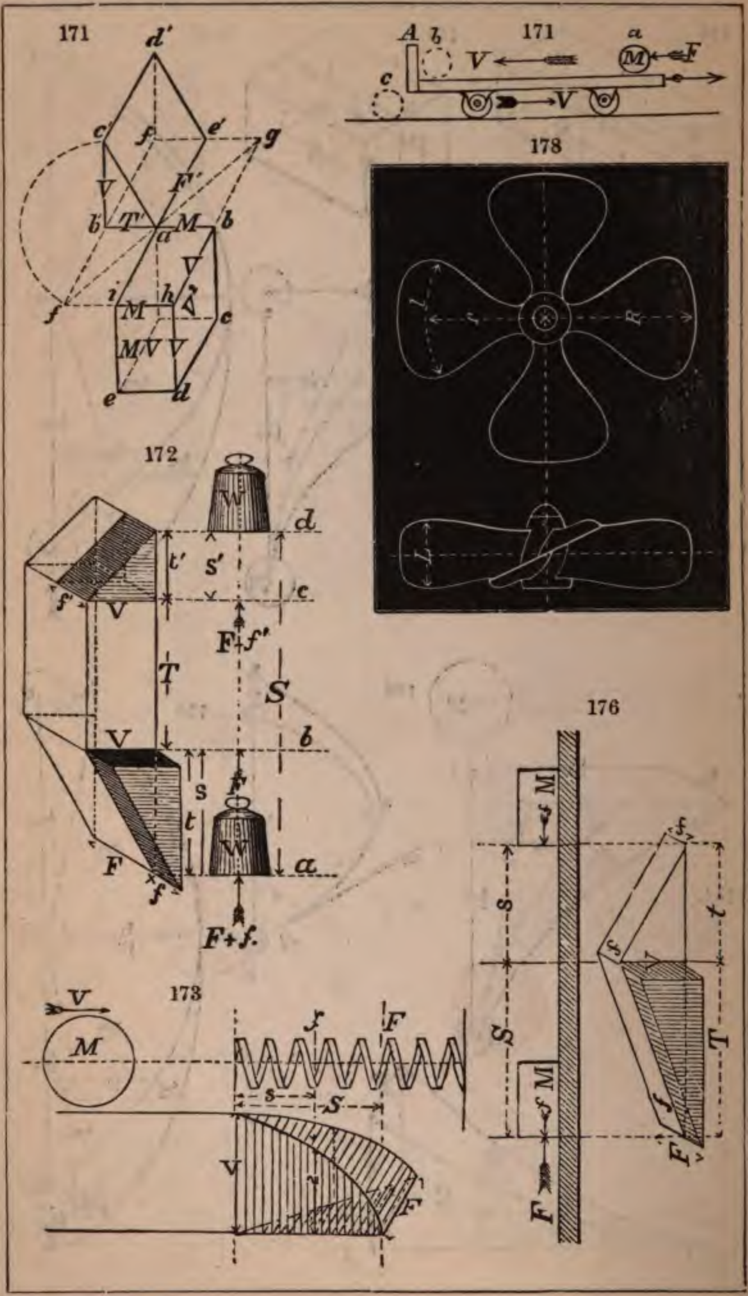
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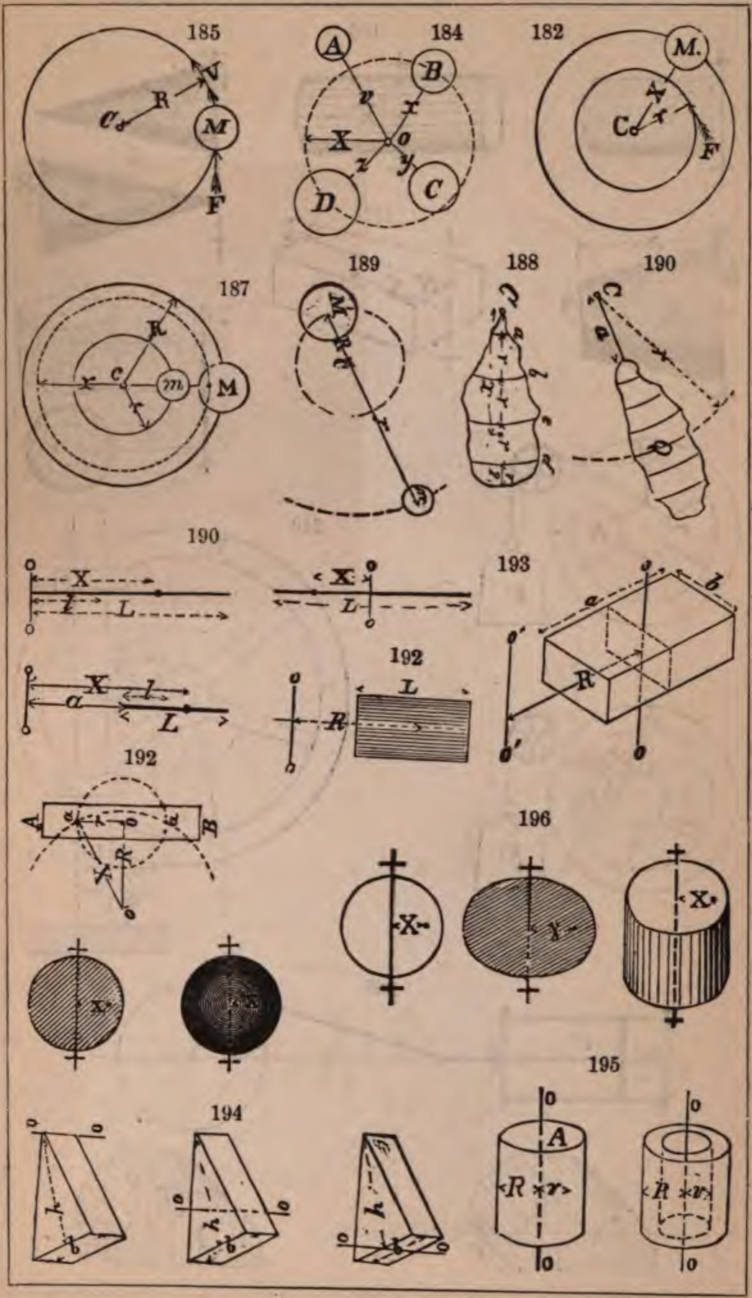
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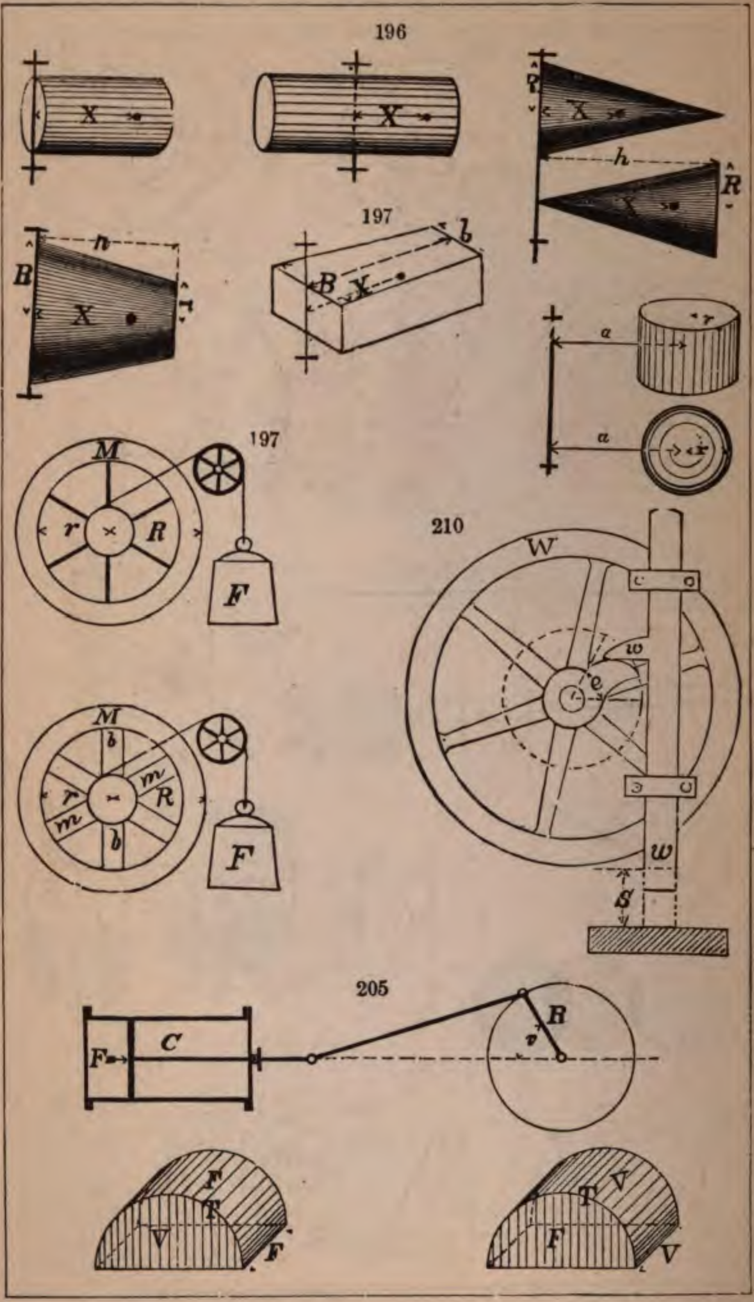
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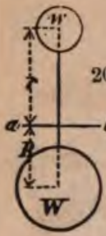

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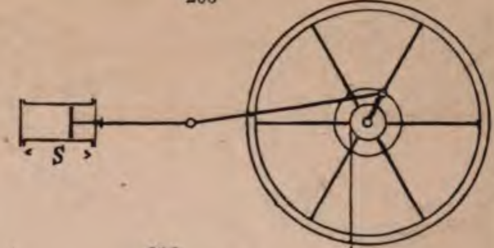

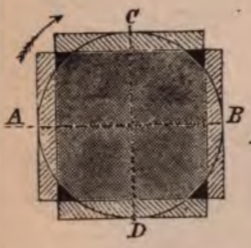

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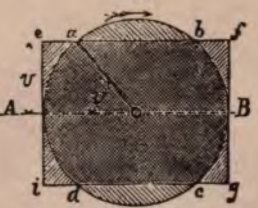

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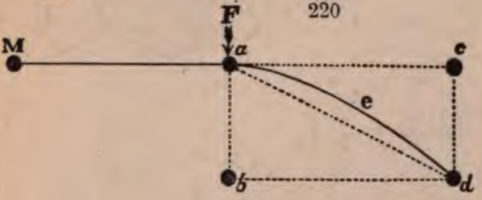

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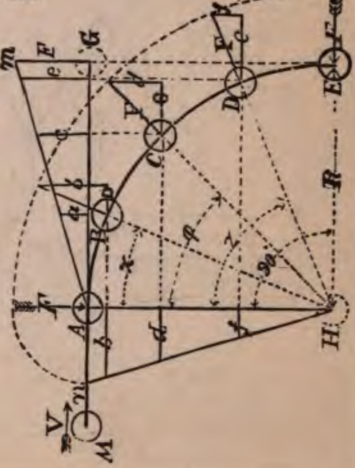

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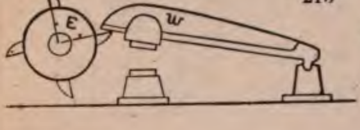

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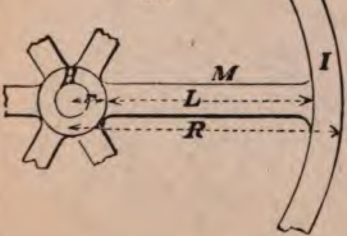

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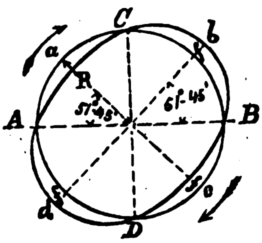

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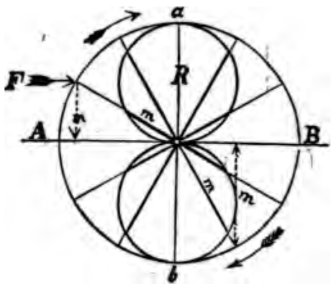

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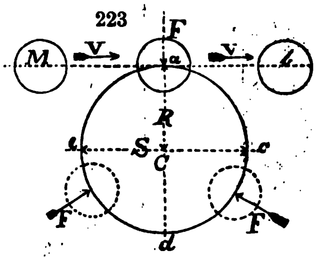

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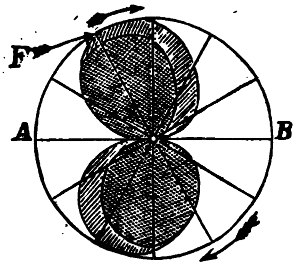

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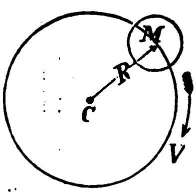

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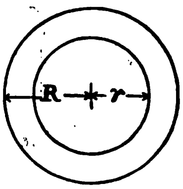

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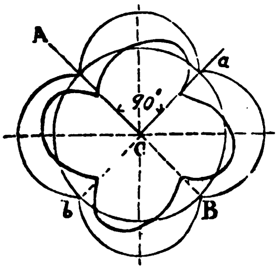

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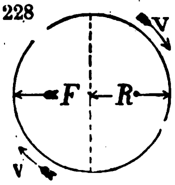

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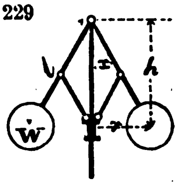

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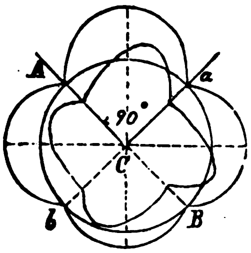

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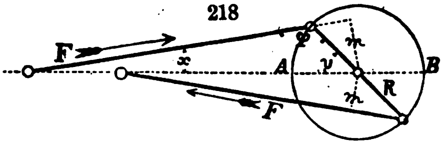

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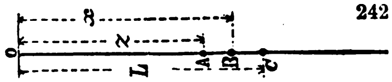

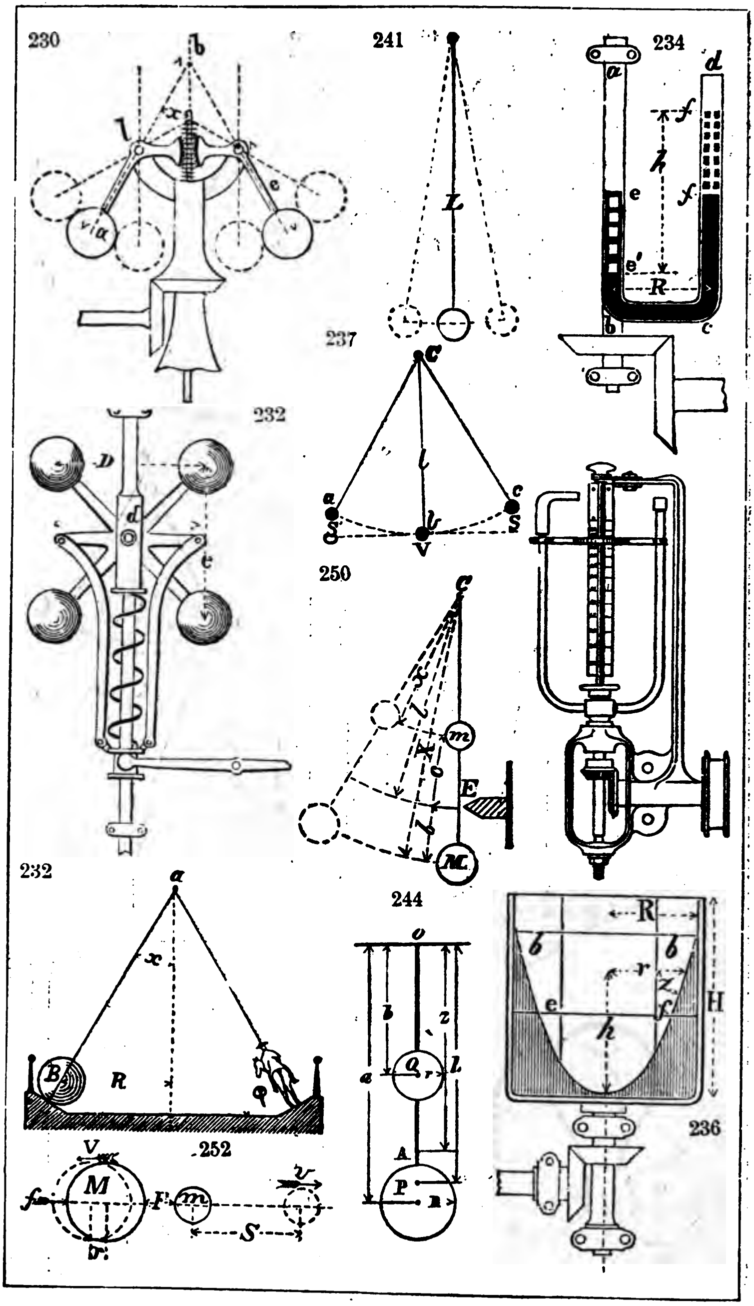
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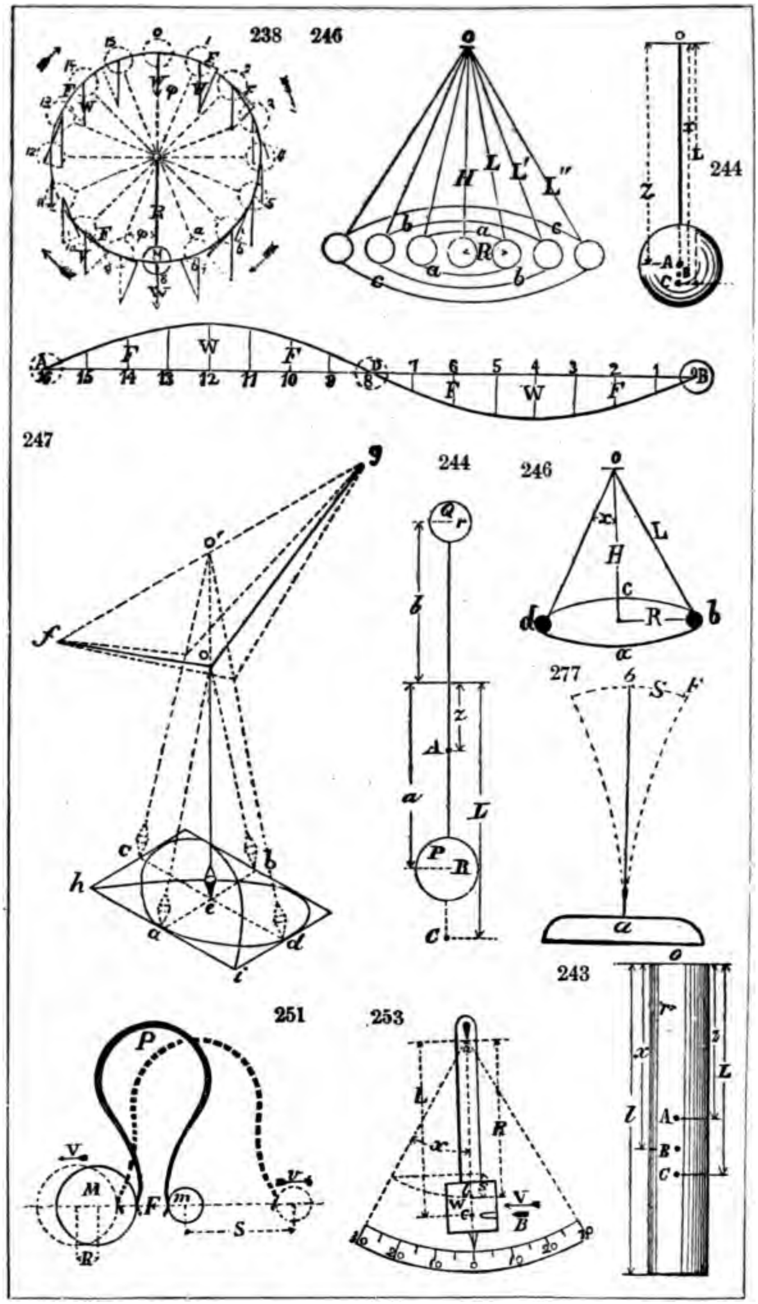
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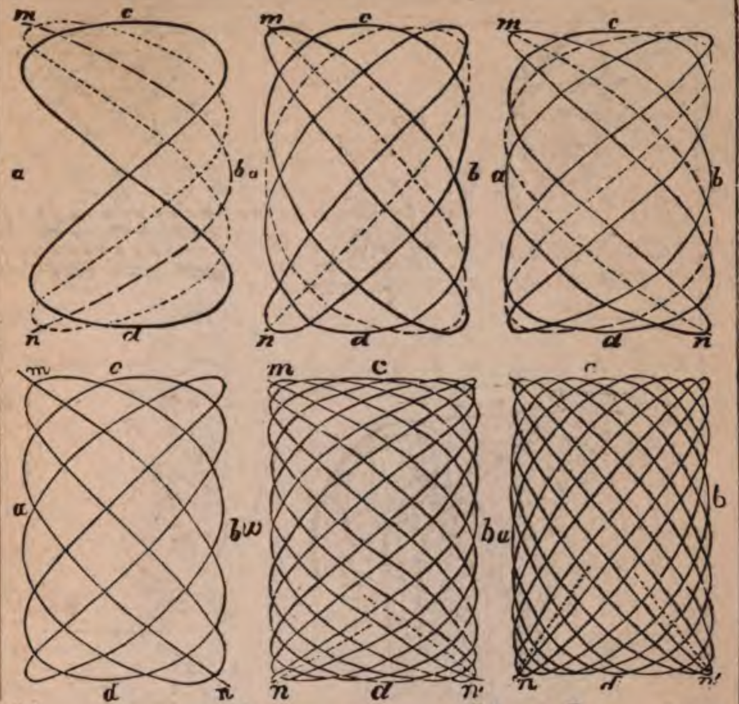

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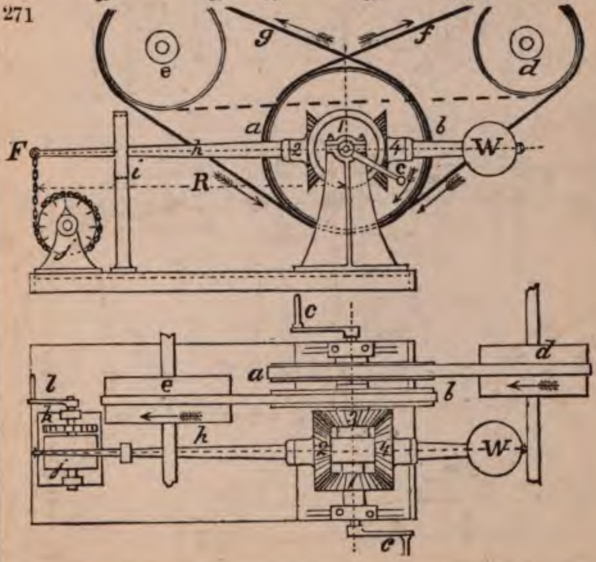

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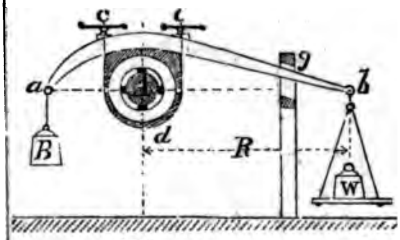

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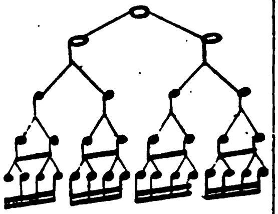

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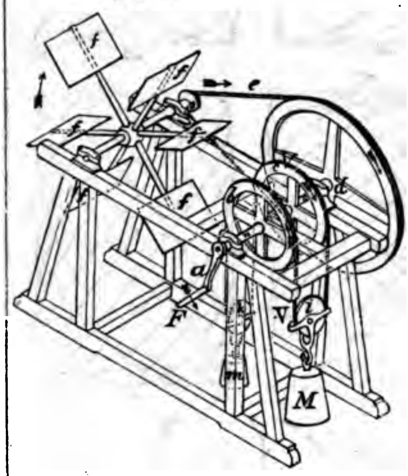

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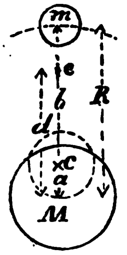

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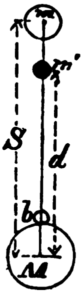

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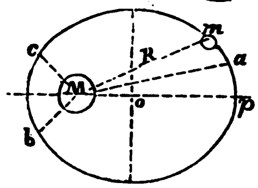

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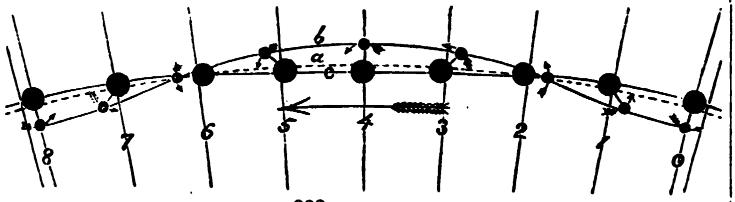

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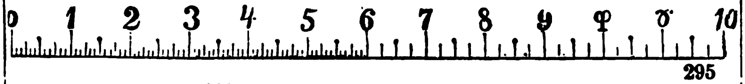

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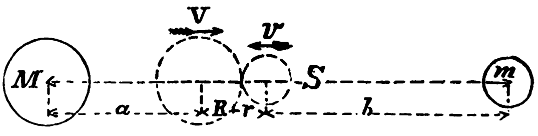

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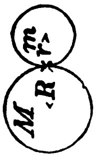

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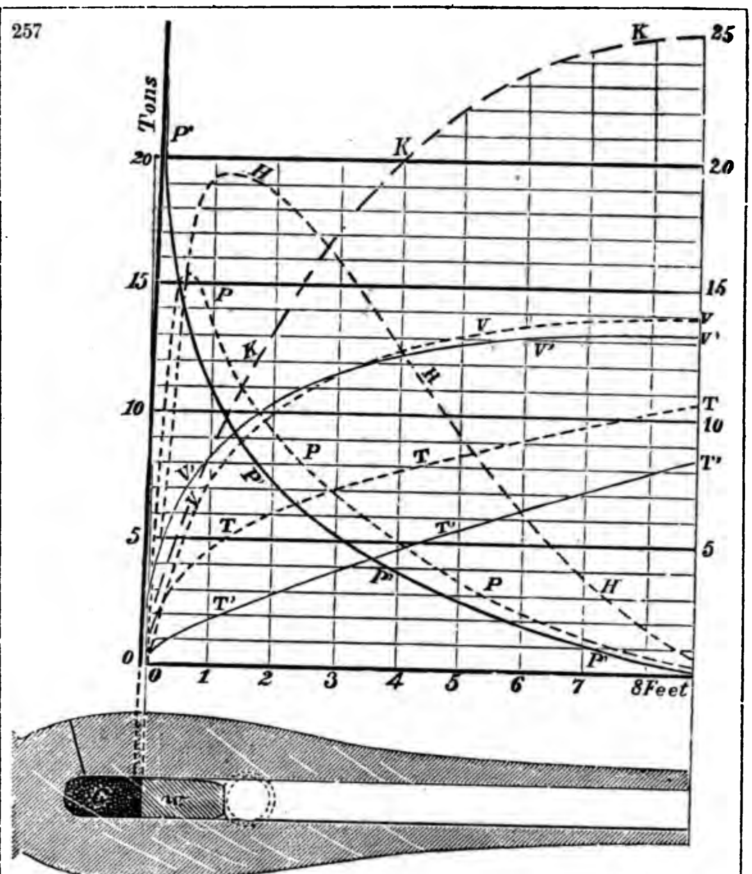

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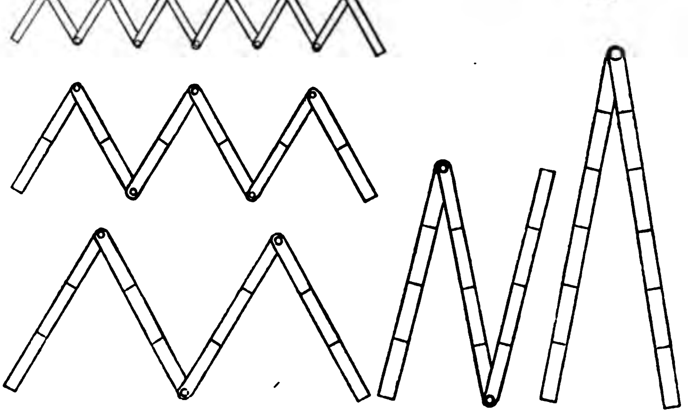

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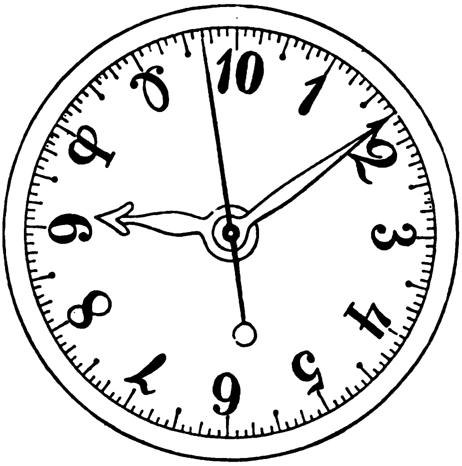

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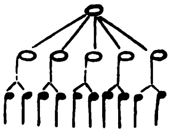

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